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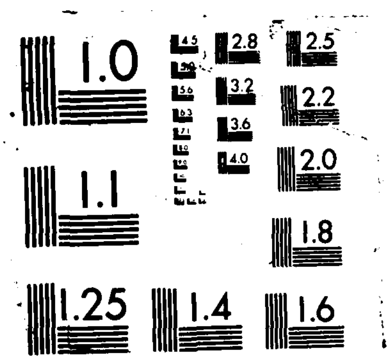
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THE h-p VERSION OF THE FINITE ELEMENT METHOD FOR PROBLEMS WITH NONHOMOGENEOUS ESSENTIAL BOUNDARY CONDITION

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BN-1069

September 1987

INSTITUTE FOR PHYSICAL SCIENCE AND TECHNOLOGY

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AD-A187356

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER BN-1069	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) The h-p Version of the Finite Element Method for Problems with Nonhomogeneous Essential Boundary Condition		5. TYPE OF REPORT & PERIOD COVERED Final life of the contract
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) I. Babuska[1] and B. Q. Guo		8. CONTRACT OR GRANT NUMBER(s) [1]ONR N00014-85-K-0169
9. PERFORMING ORGANIZATION NAME AND ADDRESS University of Maryland Institute for Physical Science and Technology College Park, MD 20742		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Department of the Navy Office of Naval Research Arlington, VA 22217		12. REPORT DATE September 1987
		13. NUMBER OF PAGES 81
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report)
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release: distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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S/N 0102-LF-014-6601

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THE h-p VERSION OF THE FINITE ELEMENT METHOD FOR PROBLEMS WITH NONHOMOGENEOUS ESSENTIAL BOUNDARY CONDITION

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[1] Supported by the Office of Naval Research under Contract N00014-85-K-0169.

[2] Supported by the National Science Foundation under Grant DMS-85-16191.

Abstract

The paper analyzes the spaces $\mathfrak{B}_{\beta}^{2}(\Omega)$ and the associated trace spaces on the boundary $\partial\Omega$. These spaces are essential in the theory of the h-p version of the finite element method. The h-p version for the problem with nonhomogeneous essential and natural boundary conditions is analyzed. Numerical experimentation is presented.

1. Introduction

There are three versions of the finite element method: the h-version, the p-version and the h-p version. The h-version is the standard one, where the degree p of the elements is fixed, usually on low levels, typically $p = 1,2$ and the accuracy is achieved by properly refining the mesh. The p-version, in contrast fixes the mesh and achieves the accuracy by increasing the degree p of the elements uniformly or selectively. The h-p version is a combination of both.

The standard h-version has been thoroughly investigated theoretically and computationally. The literature here is overwhelming. To date there are over two hundred monographs and conference proceedings [18] and new monographs and proceedings are continuously appearing. There are many programs of research and commercial type available (e.g. see [18]).

The p and h-p version is a new development and it is very successfully used for solving elliptic equations, especially in the field of computational mechanics. The first theoretical results were published in 1981 (see [2],[10]). There is only one commercial code based on the p and h-p version of the finite element, the program PROBE of NOETIC Technologies (St. Louis, MO). PROBE deals with two-dimensional elasticity, stationary heat problems and thermoelasticity problems. The code for the three-dimensional problems will be released in 1988. PROBE presently is utilizing $1 \le p \le 8$. There is also commercial code FIESTA for solving three-dimensional elasticity problems using $1 \le p \le 4$. A research code STRIPE developed by Aeronautical Research Institute of Sweden has the p and h-p version features for three-dimensional problems and is using $2 \le p \le 12$.

For the survey of the today's state of the art and recent progress we refer to [1],[2],[8],[14],[19] where also additional references can be found.

The success of the h-p version is, among others, based on the fact that the elliptic problems of the structural mechanics are usually characterized by piecewise analytic data (boundary, coefficients, boundary conditions). This implies then that the exact solution is analytic (or piecewise analytic) with singular behavior of precise character in the a-priori known areas as for example in the neighborhood of the corners of the domain. We have shown in [4],[5] that this class of solutions can be very accurately described in the frame of countably normed spaces. We have denoted this space by $\mathcal{B}^2_\beta(\Omega)$. If the solution belongs to the this class then we have shown in [6],[13] that the finite element solution converges exponentially.

The present paper elaborates on the characterization of trace spaces of the function $u \in \mathcal{B}^2_\beta(\Omega)$ and gives precisely verifiable necessary and sufficient conditions for the input data (Dirichlet and Neumann, conditions, right hand side) which guarantee that the solution belongs to $\mathcal{B}^2_\beta(\Omega)$. In the previous paper we did address the h-p version for the problems where the essential (Dirichlet) boundary conditions could be satisfied exactly by the finite element solution. In the present paper we design and analyze the way how to deal with nonhomogeneous essential boundary conditions in the full generality. We show that the performance of the method is the same for general essential conditions as for the natural ones. In section 2 we give the preliminaries and basic definitions. Section 3 defines the model problem of second order elliptic partial differential equations. Section 4 introduces the spaces of traces of $u \in \mathcal{B}^2_\beta(\Omega)$ on Γ. It shows also that the function in the trace spaces can be extended into $\mathcal{B}^2_\beta(\Omega)$. This section gives some of the major results of the paper. Section 5 defines the finite element method, its h-p version, characterizes the meshes and elements under consideration and defines how to deal with nonhomogeneous boundary conditions. Section 6 is analyzing the convergence of the method and proves that the rate of convergence is exponential. Finally, Section 7 brings numerical examples which show that the theoretical results having an asymptotic character are applicable in the wide range of practical accuracy.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^2$, $(x_1,x_2) = x$ be a simply connected, bounded domain with the boundary $\partial\Omega = \Gamma = \bigcup_{i=1}^{M} \bar{\Gamma}_i$. Γ_i are analytic simple arcs called edges,

$$\bar{\Gamma}_i \in \{(\varphi_i(\xi),\psi_i(\xi)) | \xi \in \bar{I} = [-1,1]\}$$

where $\varphi_i(\xi),\psi_i(\xi)$ are analytic functions on $\bar{I}$ and $|\varphi_i'(\xi)|^2 + |\psi_i'(\xi)|^2 \geq \alpha_i > 0$. By Γ_i we denote the open arc, i.e., the image of $I = (-1,1)$. Let A_i, $i = 1,\ldots,M$ be the vertices of Ω and $\Gamma_i = A_iA_{i+1}$, i.e., the edge Γ_i is linking the vertices A_i and A_{i+1}. For simplicity we will also write $A_1 = A_{M+1}$. An example of the domain Ω under consideration is given in Figure 2.1.

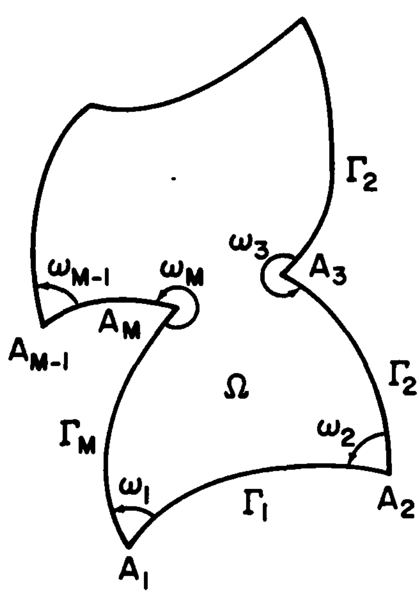

Figure 2.1. The scheme of the domain.

By ω_i, $i = 1,\ldots,M$ we denote the internal angles of Ω at A_i. We shall assume that $0 < \omega_i \leq 2\pi$. We will also consider the case when two edges coincide. Then we understand them in a "two sided" sense. If all edges are straight lines then we will call the domain Ω a straight polygon. Otherwise we will speak about a curvilinear polygon. If $0 < \omega_i < 2\pi$, $i = 1,\dots,M$, we will speak about a Lipschitzian domain. Let us assume that $\Gamma = \Gamma^{(0)} \cup \Gamma^{(1)}$ where $\Gamma^{(0)} = \bigcup_{i\in Q} \bar{\Gamma}_i$, $\Gamma^{(1)} = \Gamma - \Gamma^{(0)}$, $\bar{\Gamma}^{(1)} = \bigcup_{i\in Q'} \bar{\Gamma}_i$, where Q is some subset of the set $\{1,2,\dots,M\} = \mathcal{M}$ and $Q' = \mathcal{M} - Q$.

We have assumed for simplicity that Ω is a simply connected domain. The results we are presenting here are also valid when Ω is n-connected, bounded domain and its boundary is composed by n-curves.

Denote $I = \{x \mid -1 < x < 1\}$, we also will write $I = \{x_1, x_2 \mid -1 < x_1 < 1,\ x_2 = 0\} \subset \mathbb{R}_2$ when no misunderstanding could occur.

By $L_2(\Omega)$, $L_p(\Omega)$, $L_2(I)$, $L_p(I)$ the usual spaces of p-integrable, $1 < p < \infty$, functions on Ω or I are denoted. By $H^m(\Omega)$, $H^m(I)$, $m \ge 0$ an integer we denote the usual Sobolev space of functions with square integrable derivatives of order $\le m$ on Ω (respectively I). The space $H^m(\Omega)$ is furnished with the usual norm

$$\|u\|^2_{H^m(\Omega)} = \sum_{0 \le |\alpha| \le m} \|D^\alpha u\|^2_{L_2(\Omega)}$$

where $\alpha = (\alpha_1, \alpha_2)$, $\alpha_i \ge 0$ integer, $i = 1,2$, $|\alpha| = \alpha_1 + \alpha_2$ and

$$D^\alpha u = \frac{\partial^\alpha u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} = u_{x_1^{\alpha_1} x_2^{\alpha_2}}.$$

Further we let

$$|u|_{H^m(\Omega)} = \| \, |D^m u| \, \|_{L_2(\Omega)},$$

$$|D^m u|^2 = \sum_{|\alpha| = m} |D^\alpha u|^2.$$

As usual we shall write $H^0(\Omega) = L_2(\Omega)$,

$$H_0^1(\Omega) = \{u \in H^1(\Omega) | u = 0 \text{ on } \Gamma^{(0)}\}.$$

In the analogous way we define $H^m(I)$ with $D^k u = u^{(k)} = \frac{d^k u}{dx^k}$.

By $r_j(x) = \operatorname{dist}(x, A_j) = |x - A_j|$, $x \in \Omega$, $j \in \mathcal{M}$ we shall denote the Euclidean distance between the point x and the vertex A_j, $\hat{r}_1(x) = |x+1|$, $\hat{r}_2(x) = |x-1|$, $x \in I$. Let $\beta = (\beta_1, \ldots, \beta_M)$ (respectively $\beta = (\beta_1, \beta_2)$) be an M-tuple of real numbers $0 < \beta_i < 1$, $i = 1, \ldots, M$. We will write $\alpha_1 < \beta < \alpha_2$ (respectively $\bar{\beta} < \beta$) if $\alpha_1 < \beta_i < \alpha_2$ (respectively $\bar{\beta}_i < \beta_i$), $i = 1, \ldots, M$. For any k integer we shall write $\beta + k = (\beta_1 + k, \ldots, \beta_M + k)$ and

$$\Phi_{\beta+k}(x) = \prod_{i=1}^{M} |r_i(x)|^{\beta_i + k}, \quad x \in \Omega$$

and

$$\hat{\Phi}_{\beta+k}(x) = \prod_{i=1}^{2} |\hat{r}_i(x)|^{\beta_i + k}, \quad x \in I.$$

By $C^j(\Omega)$, $C^j(\bar{\Omega})$, $C^j(I)$, $C^j(\bar{I})$, $j \geq 0$ integer we will denote the set of all functions with continuous j-derivatives on Ω, $\bar{\Omega}$, I, $\bar{I}$, furnished with the usual norm $\|\cdot\|_{C^j(\Omega)}$, $\|\cdot\|_{C^j(I)}$. Let $H_\beta^{m,\ell}(\Omega)$, $m \geq \ell \geq 0$ integers be the completion of the set of all infinitely differentiable functions under the norm

$$\|u\|^2_{H_\beta^{m,\ell}(\Omega)} = \|u\|^2_{H^{\ell-1}(\Omega)} + \sum_{\substack{|\alpha|=\ell \\ k=\ell}}^{k=m} \|\Phi_{\beta+k-\ell} |D^{|\alpha|} u| \|^2_{L_2(\Omega)} \quad \text{for } \ell \geq 1,$$

$$\|u\|^2_{H_\beta^{m,0}(\Omega)} = \sum_{\substack{|\alpha|=k \\ k=0}}^{k=m} \|\Phi_{\beta+k} |D^{\alpha} u| \|^2_{L_2(\Omega)}.$$

If $m = \ell = 0$ we shall write $H_\beta^{0,0} = L_\beta(\Omega)$. Analogously as before we define

$$|u|^2_{H_\beta^{\ell,\ell}(\Omega)} = \sum_{|\alpha|=\ell} \|\Phi_\beta |D^\alpha u|\|^2_{L_2(\Omega)}.$$

In the similar way $H_\beta^{m,\ell}(I)$ is defined

$$\|u\|^2_{H_\beta^{m,\ell}(I)} = \|u\|^2_{H^{\ell-1}(I)} + \sum_{k=\ell}^{k=m} \|\hat{\Phi}_{\beta+k-\ell} |D^\beta u|\|^2_{L_2(I)} \quad \text{for} \quad \ell \geq 1,$$

$$\|u\|^2_{H_\beta^{m,0}(I)} = \sum_{k=0}^{k=m} \|\hat{\Phi}_{\beta+k} |D^\alpha u|\|^2_{L_2(I)}.$$

Further we introduce the space $\mathcal{B}_\beta^\ell(\Omega)$, $\ell \geq 0$ integer which will play an important role in this paper:

$$\mathcal{B}_\beta^\ell(\Omega) = \{u|u \in H_\beta^{k,\ell}(\Omega), \text{ any } k \geq \ell, \|\Phi_{\beta+k-\ell}|D^\alpha u|\|_{L_2(\Omega)} \leq Cd^{k-\ell}(k-\ell)!, |\alpha| = k, C > 0, d \geq 1 \text{ independent of } k\}.$$

If we wish to underline the dependence on d we will write $\mathcal{B}_{\beta,d}^\ell(\Omega)$. Analogously for $\ell \geq 0$ integer

$$\mathcal{B}_\beta^\ell(I) = \{u|u \in H_\beta^{k,\ell}(I), \text{ any } k \geq \ell, \|\hat{\Phi}_{\beta+k-\ell}u^{(k)}\|_{L_2(I)} \leq Cd^{k-\ell}(k-\ell)!, C > 0, d \geq 1 \text{ independent of } k\}.$$

Further for $j = 1,2$,

$$\mathcal{C}_\beta^j(\Omega) = \{u \in H_\beta^{j,j}(\Omega) | |D^\alpha u|(x) \leq Cd^k k! |\Phi_{k+\beta-j+1}(x)|^{-1}$$
$$|\alpha| = k = j-1, j, \ldots, C > 0, d \geq 1 \text{ independent of } k\},$$

$$\mathfrak{C}^j_\beta(I) = \{u \in H^{j,j}_\beta(I) \mid |u^{(k)}| < C|\hat{\Phi}_{k+\beta-j+1/2}(x)|^{-1} d^k k!$$
$$k > j-1,\ldots, C > 0,\ d \geq 1 \text{ independent of } k\}.$$

Let $\gamma \in \bigcup_{i \in \rho \subset \mathcal{M}} \bar{\Gamma}_i$. Then we define $H^{k-1/2}(\gamma)$ (respectively $H^{k-1/2,\ell-1/2}_\beta(\gamma)$, $k \geq \ell$), $k \geq \ell \geq 1$ integers as follows: for any $\varphi \in H^{k-1/2}(\gamma)$ (respectively $H^{k-1/2,\ell-1/2}_\beta(\gamma)$) there exists $f \in H^k(\Omega)$, (respectively $H^{k,\ell}_\beta(\Omega)$) such that $f|_\gamma = \varphi$. We define then

$$\|\varphi\|_{H^{k-1/2}(\Gamma)} \quad (\text{respectively } \|\varphi\|_{H^{k-1/2,\ell-1/2}_\beta(\gamma)})$$
$$= \inf_{f|_\gamma = \varphi} \|f\|_{H^k(\Omega)} \quad (\text{respectively } \|f\|_{H^{k,\ell}_\beta(\Omega)}).$$

By $\mathfrak{B}^{\ell-1/2}_\beta(\gamma)$, $\ell \geq 1$, we will denote the set of the traces on γ of functions from the space $\mathfrak{B}^\ell_\beta(\Omega)$.

Let Γ_i be an edge of Ω, then by the assumption there exists a one to one mapping m_i of I onto Γ_i which is analytic. If Γ_0 is a straight line then we shall assume that m_i is the linear mapping. Let u be defined on Γ_i, $U(x) = u(m_i(x))$ be defined on I. Then we define

$$H^m(\Gamma_i) = \{u \mid U \in H^m(I)\}$$
$$\|u\|_{H^m(\Gamma_i)} = \|U\|_{H^m(I)}.$$

In the same way we define the spaces $H^{m,\ell}_\beta(\Gamma_i)$, $\mathfrak{B}^\ell_\beta(\Gamma_i)$, $\mathfrak{C}^\ell_\beta(\Gamma_i)$. Let us remark that, as we defined it, $\|\cdot\|_{H^m(\Gamma_i)}$ depends on the mapping m_i, i.e., it depends on the parameterization of the arc Γ_i. Nevertheless the space $H^{m,\ell}_\beta(\Gamma_i)$ does not as well as $\mathfrak{C}^\ell_\beta(\Gamma_i)$ (see Lemma 4.6) but $\mathfrak{B}^\ell_\beta(\Gamma_i)$ could be dependent on m_i. Let us state now some lemmas which will be used later.

Lemma 2.1. We have

$$H_{\beta}^{2,2}(\Omega) \subset C^{0}(\bar{\Omega})$$

with the continuous injection. □

See Lemma 7 of [3].

Lemma 2.2. Let $u \in H_{\beta}^{2,2}(\Omega)$. Then

(i)

(2.1) $$\| |D^{1}u| \Phi_{\beta-1} \|_{L_2(\Omega)} \le C \|u\|_{H_{\beta}^{2,2}(\Omega)} .$$

(ii) Let $u(A_i) = 0$, $i = 1,\dots,M$. Then

(2.2) $$\| u \Phi_{\beta-2} \|_{L_2(\Omega)} \le C \|u\|_{H_{\beta}^{2,2}(\Omega)} .$$

See Lemma 8 of [2]. □

Lemma 2.3. $\mathfrak{B}_{\beta}^{2}(\Omega) \subset \mathfrak{C}_{\beta}^{2}(\Omega)$ and $\mathfrak{C}_{\beta}^{2}(\Omega) \subset \mathfrak{B}_{\beta+\varepsilon}^{2}(\Omega)$, $0 < \beta+\varepsilon < 1$, $\varepsilon > 0$ arbitrary. □

See Theorem 2.2 and 2.3 of [6].

Lemma 2.4. Let $u \in \mathfrak{B}_{\beta}^{j}(\Omega)$, $j \ge 0$, then u is analytic on $\bar{\Omega} - \bigcup_{i=1}^{M} A_i$. □

Lemma 2.5. Let $r \ne 1$ and $F(x)$, $0 < x < \infty$ is defined by

$$F(x) = \int_0^x f(t)dt \quad \text{for} \quad r > 1$$

$$F(x) = \int_x^\infty f(t)dt \quad \text{for} \quad r < 1.$$

Then

$$\int_0^\infty x^{-r} F^2(x)\,dx \le \left(\frac{2}{|r-1|}\right)^2 \int_0^\infty x^{-r} (xf)^2\,dx. \qquad \square$$

See Theorem 330 of [16].

3. The model problem and its properties

Let Ω be the curvilinear or straight polygon and L be a strongly elliptic operator

$$L(u) = -\sum_{i,j=1}^{2} (a_{i,j}(x)u_{x_i})_{x_j} + \sum_{i=1}^{2} b_i(x)u_{x_i} + c(x)u$$

where $a_{i,j}(x) = a_{j,i}(x)$, $b_i(x)$, $c(x)$ are analytic functions on $\bar{\Omega}$ and for any $\xi_1, \xi_2 \in \mathbb{R}$ and any $x \in \Omega$ let

$$\sum_{i,j=1}^{2} a_{i,j}\xi_i\xi_j \geq \mu_0(\xi_1^2+\xi_2^2)$$

with $\mu_0 > 0$.

Let $B(u,v)$ be continuous bilinear form on $H_0^1(\Omega)\times H_0^1(\Omega)$

$$B(u,v) = \int_\Omega \left[\sum_{i,j=1}^{2} a_{i,j}u_{x_i}v_{x_j} + \sum_{i=1} b_iu_{x_i}v + cuv\right]dx.$$

We assume that

$$\inf_{\substack{\|u\|_{H^1(\Omega)}=1 \\ u\in H_0^1(\Omega)}} \sup_{\substack{\|v\|_{H^1(\Omega)}=1 \\ v\in H_0^1(\Omega)}} |B(u,v)| \geq \mu_1 > 0$$

and for any $v \in H_0^1(\Omega)$, $v \neq 0$

$$\sup_{\substack{\|u\|_{H^1(\Omega)}=1 \\ u\in H_0^1(\Omega)}} |B(u,v)| > 0.$$

Assume now that $g^{[\ell]} \in \mathcal{B}_\beta^{3/2-\ell}(\Gamma^{(\ell)})$, $\ell = 0,1$, $f \in \mathcal{B}_\beta^0(\Omega)$ and consider the boundary value problem

(3.1a) $$Lu = f \quad \text{on} \quad \Omega$$

(3.1b) $$u = g^{[0]} \quad \text{on} \quad \Gamma^{(0)}$$

(3.1c) $$\frac{\partial u}{\partial n_c} = g^{[1]} \quad \text{on} \quad \Gamma^{(1)}$$

where we denoted by n_c the conormal in the usual sense. The solution of our problem is understood in the usual sense. Then we have

<u>Theorem 3.1</u>. There exists unique solution $u_0 \in H^1(\Omega)$ of the problem (3.1). See Lemma 3.1 of [4].

Let us mention some theorems addressing regularity of the solution u_0.

<u>Theorem 3.2</u>. There exists $0 \le \bar{\beta}_i < 1$, $i = 1,\dots,M$ depending in the problem (i.e., operator L, ω_i, etc.), such that if $f \in \mathcal{B}^0_\beta(\Omega)$, $g^{[\ell]} \in \mathcal{B}^{3/2-\ell}_\beta(\Gamma^{(\ell)})$, $\ell = 0,1$, $\bar{\beta} < \beta < 1$ then $u_0 \in \mathcal{B}^2_\beta(\Omega))$.

Proof is given in [3].

<u>Theorem 3.3</u>. Let Ω be a (curvilinear) polygon (instead of straight polygon as in Theorem 3.2) and let then assumptions of Theorem 3.2. hold. Then $u_0 \in \mathcal{C}^2_\beta(\Omega)$.

Proof of the theorem is given in [6].

We have seen in [6], [13] (see also sections 5 and 6) that when the solution u of the problem 3.1 belongs to the class $\mathcal{B}^2_\beta(\Omega)$ then the h-p version of the finite element method converges exponentially.

Theorems 3.1 and 3.2 show that it is important to develop practical characterizations of spaces $\mathcal{B}^{3/2-\ell}_\beta(\Gamma)$, $\ell = 0,1$, which can be easily used in concrete cases to verify whether the input data, i.e., $g^{[\ell]}$ belong to the desired space. We will elaborate on it in the next section.

4. Traces and extensions of weighted Sobolev spaces. Characterization of the spaces $\mathfrak{B}_\beta^{3/2-\ell}(\Gamma)$

In this section we will elaborate on the characterization of the space $\mathfrak{B}_\beta^{3/2-\ell}(\Gamma)$, $\ell = 0,1$ which leads to an easy verification in the concrete cases of applications.

Lemma 4.1. Let $\beta = (\beta_1,\beta_2)$, $0 < \beta < 1/2$ and $g \in H_\beta^{1,1}(I)$. Then

(i) $g \in C^0(\bar{I})$ and $\|g\|_{C^0(\bar{I})} \le C\|g\|_{H_\beta^{1,1}(I)}$

(ii) $|g(x)-g(-1)| \le C\hat{\Phi}_{1/2-\beta}(x)\|g\|_{H_\beta^{1,1}(I)}$

$|g(x)-g(1)| \le C\hat{\Phi}_{1/2-\beta}(x)\|g\|_{H_\beta^{1,1}(I)}$

where C is a constant independent of $g(x)$ (but depends on β).

Proof. Obviously

$$|g(x)-g(t)| \le \left|\int_t^x g'(\tau)d\tau\right|$$

$$(4.1) \qquad \le \left[\int_t^x g'^2(\tau)\hat{\Phi}_\beta^2(\tau)d\tau\right]^{1/2}\left[\int_t^x (\hat{\Phi}_\beta(\tau))^{-2}d\tau\right]^{1/2}$$

$$\le \|g\|_{H_\beta^{1,1}(I)}\left[\int_t^x (\hat{\Phi}_\beta(\tau))^{-2}d\tau\right]^{1/2}$$

which shows that g is continuous on $\bar{I}$. Using the imbedding theorem on $(-1/2,1/2) = I'$ we have

$$(4.2) \qquad |g(0)| \le C\|g\|_{H^1(I')} \le C\|g\|_{H_\beta^{1,1}(I)}$$

and we get immediately

$$\|g\|_{C^0(I)} \le C\|g\|_{H_\beta^{1,1}(I)}.$$

Further (4.1) immediately leads to (ii). □

Lemma 4.2. Let $\beta = (\beta_1,\beta_2)$, $1/2 < \beta < 1$ and $g \in H^{2,2}_\beta(I)$. Then

(i) $g \in C^0(I)$ and $\|g\|_{C^0(I)} \le C\|g\|_{H^{2,2}_\beta(I)}$

(ii) $|g(x)-g(-1)| \le C\hat{\Phi}_{3/2-\beta}(x)\|g\|_{H^{2,2}_\beta(I)}$

$|g(x)-g(1)| \le C\hat{\Phi}_{3/2-\beta}(x)\|g\|_{H^{2,2}_\beta(I)}$

where C is a constant independent of $g(x)$.

Proof. Using (4.1) we get

$$
\begin{aligned}
|g(x)-g(t)| &\le \left|\int_t^x g'(\tau)d\tau\right| \\
&\le \left[\int_t^x g'^2(\tau)\hat{\Phi}^{-2}_{1-\beta}d\tau\right]^{1/2}\left[\int_t^x \hat{\Phi}^2_{1-\beta}(\tau)d\tau\right]^{1/2} \\
&\le \|g'\hat{\Phi}^{-1}_{1-\beta}\|_{L_2(I)}\left[\int_t^x \hat{\Phi}^2_{1-\beta}(\tau)d\tau\right]^{1/2}
\end{aligned}
$$

and

$$
\begin{aligned}
\|g'\hat{\Phi}^{-1}_{1-\beta}\|_{L_2(I)} &\le \|(g'-g'(0))\hat{\Phi}^{-1}_{1-\beta}\|_{L_2(I)} \\
&\quad + |g'(0)|\|\hat{\Phi}^{-1}_{1-\beta}\|_{L_2(I)} \\
&\le C[|g'(0)| + \|g''\hat{\Phi}_\beta\|_{L_2(I)}] \\
&\le C\|g\|_{H^{2,2}_\beta(I)}.
\end{aligned}
$$

In the last inequality we used Lemma 2.5 and the fact that $1/2 < \beta < 1$. The lemma now follows immediately. □

Lemma 4.3. Let $g \in \mathcal{B}^1_{\beta,d}(I)$, $0 < \beta < 1$. Then for $k \ge 1$

$$|g^{(k)}(x)| \le C(\hat{\Phi}_{k-1/2+\beta}(x))^{-1}(d_1)^k k!$$

where $\gamma > 1$ is independent of g,k,d, and C depends on β, but is independent of g,k.

<u>Proof</u>. Let $I' = (-1/2,1/2)$. Then for any $k \ge 1$ we have

$$\|g^{(k)}\|_{H^1(I')} \le C(\hat{\Phi}(1/2))^{-k-\bar{\beta}} k! d^k$$

where $\bar{\beta} = \max(\beta_1,\beta_2)$. Hence by the imbedding theorem

$$|g^{(k)}(0)| \le C d_1^k k!$$

where $d_1 \ge \gamma d$, $\gamma > \hat{\Phi}^{-1}(1/2) > 1$. Further, for $k \ge 1$

$$|g^{(k)}(x)| \le |g^{(k)}(0)| + \left|\int_0^x g^{(k+1)}(t)dt\right|$$

$$\le |g^{(k)}(0)| + \left[\int_0^x (g^{(k+1)}(t))^2 \hat{\Phi}^2_{\beta+k}(t)dt\right]^{1/2}$$

$$\left[\int_0^x \hat{\Phi}^{-2}_{\beta+k}(t)dt\right]^{1/2}$$

$$\le C d_1^k k![1+\hat{\Phi}^{-1}_{\beta+k-1/2}(x)]$$

$$\le C(d_1)^k k!(\hat{\Phi}_{k-1/2+\beta}(x))^{-1}.$$

<u>Corollary 4.4</u>. Let $g \in \mathcal{B}^1_\beta(I)$, $0 < \beta < 1$. Then $g \in \mathcal{C}^1_\beta(I)$.

<u>Corollary 4.5</u>. Let $g \in \mathcal{B}^2_\beta(I)$, $0 < \beta < 1$. Then for $k \ge 2$

$$|g^{(k)}(x)| \le C(\hat{\Phi}_{k-3/2+\beta}(x))^{-1} d_1^k k!$$

and $g \in \mathcal{C}^2_\beta(I)$.

<u>Lemma 4.6</u>. Let $\xi = m(x)$ be a one to one map of $\bar{I}$ onto $\bar{I}$, $m(x)$ be analytic on $\bar{I}$ and $|m'(x)| > 0$, $x \in \bar{I}$. Assume that $g \in \mathfrak{C}^j_\beta(I)$, $j = 1,2$, and define $v(x) = g(m(x))$. Then $v \in \mathfrak{C}^j_\beta(I)$, $j = 1,2$.

<u>Proof</u>. Because $m(x)$ is analytic on $\bar{I}$ it can be extended into the complex plane $\mathbb{C}$ on $I_\delta = \{z = x+iy \mid -1-\delta < x < 1+\delta, |y| < \delta\}$, $\delta > 0$, $m(z)$ is a one to one mapping of $\bar{I}_\delta$ onto $\bar{I}^*_\delta \supset I_{\delta'}$, $\delta' > 0$ and $|m'(z)| > \alpha_0 > 0$, $z \in \bar{I}_\delta$. Let now $j = 1$ and $x_0 \in I$. Then for $k \ge 1$

$$|g^{(k)}(x_0)| \le C(\hat{\Phi}_{k-1/2+\beta}(x_0))^{-1} d_1^k k!$$

and the series

$$g'(x) = \sum_{k=0}^{\infty} g^{(k+1)}(x_0)(x-x_0)^k \frac{1}{k}!$$

is absolutely convergent for $|x-x_0| \le \alpha \dfrac{\hat{\Phi}(x_0)}{d_1}$, $\alpha < 1$. Hence also

$$g'(z) = \sum_{k=0}^{\infty} g^{(k+1)}(x_0)(z-x_0)^k \frac{1}{k}!$$

converges and $|g'(z)| \le C\hat{\Phi}^{-1}_{\beta+1/2}(x_0)$ for $|z-x_0| < \alpha \dfrac{\hat{\Phi}(x_0)}{d_1}$ where C is independent of x_0. Hence $g(z)$ is a holomorphic function and $v(z) = g(m(z))$ is holomorphic, too. Using Cauchy theorem we get immediately that for $k \ge 1$

$$|v^{(k)}(x)| \le Cd_2^k \hat{\Phi}^{-1}_{k-1/2+\beta}(x) k!.$$

Obviously $v(x) \in H^{1,1}_\beta(I)$. In quite a similar way we prove the statement for $j = 2$.

Remark 4.1. Lemma 4.6 shows that the space $\mathfrak{C}^j_\beta(I)$ is invariant with respect to an analytic mapping. Using the formula of the n^{th} derivative of a composite function (see formula 0.430 of [15]) we can also show that $\mathfrak{B}^j_\beta(I)$ is invariant space with respect to an analytic mapping $m(x)$ as in Lemma 4.6.

Let Γ be an analytic arc. Then we could define the spaces $\mathfrak{C}^j(\Gamma)$ and $\mathfrak{B}^j_\beta(\Gamma)$ with respect to the length instead as we did in section 2 using a specific mapping. These two definitions are then equivalent by lemma 4.6. and Remark 4.1.

Lemma 4.7. Let $M(x)$, $x \in \mathbb{R}^2$, $M(x) = (M_1(x), M_2(x))$ is a one to one mapping of $\bar{\Omega}$ onto $\bar{\tilde{\Omega}}$ and $|J^{-1}| \le \mu$ on $\bar{\Omega}$, where J is the Jacobian of the mapping. Assume that $M(x)$ can be analytically extended on $\bar{\Omega}_\delta = \{x \in \mathbb{R}^2 | \mathrm{dist}(x,\Omega) \le \delta\}$ so that it is one to one mapping of $\bar{\Omega}_\delta$ onto $\bar{\Omega}^*$, $\Omega^* \supset \bar{\Omega}$. Let $u \in \mathfrak{C}^j_\beta(\Omega)$, $j = 1,2$, $v(M(x)) = u(x)$. Then $v \in \mathfrak{C}^j_\beta(\tilde{\Omega})$. The proof is quite analogous as of the Lemma 4.6 only we have to apply the theory of two complex variables.

Lemma 4.8. Let $g \in \mathfrak{C}^j_\beta(I)$, $0 < \beta < 1$, $j = 1,2$. Then

$$g \in \mathfrak{B}^j_{\bar{\beta}}(I), \quad 0 < \bar{\beta} < 1,$$

$$\bar{\beta} = \beta + \varepsilon, \quad \varepsilon > 0 \quad \text{arbitrary.}$$

Proof. Let us consider only the case $j = 1$. The case $j = 2$ is analogous. Because for $k > 1$

$$|g^{(k)}(x)| \le Cd^k k! (\hat{\Phi}_{k+\beta-1/2}(x))^{-1}$$

we get

$$\int_{-1}^{1} (g^{(k)}(x))^2 \hat{\Phi}^2_{k+\beta-1}(x)dx$$

$$\le Cd^{2k}(k!)^2 \int_{-1}^{1} \hat{\Phi}^2_{\bar{\beta}-\beta-1/2}(x)dx$$

$$\le C(\varepsilon)d^{2k}(k!)^2 .$$

We see that Lemma 2.3 has a completely analogous version for the relation between $\mathfrak{B}^2_{\beta}(I)$ and $\mathfrak{C}^2_{\beta}(I)$.

<u>Theorem 4.1</u>. Let $u \in H^{k+2,2}_{\beta}(\Omega)$, $k \ge 0$ and Γ_i be a straight line edge of Ω and $u|_{\Gamma_i} = g_i$. Then

(i) For $1/2 < \beta_i, \beta_{i+1} < 1$ and $k \ge 0$

$$g_i \in H^{k+1,1}_{\hat{\beta}_i}(\Gamma_i),\ \hat{\beta}_i = (\hat{\beta}_{i,1}, \hat{\beta}_{i,2})$$

$$\hat{\beta}_{i,j} > 0,\ \hat{\beta}_{i,j} \in (\beta_{i+j-1}-1/2, 1),\ j = 1,2$$

and

$$\|g_i\|_{H^{k+1,1}_{\hat{\beta}_i}(\Gamma_i)} \le Cd^k \|u\|_{H^{k+2,2}_{\beta}(\Omega)}$$

with C independent of k and $d \ge 1$.

(ii) For $0 < \beta_i, \beta_{i+1} < 1/2$, $k \ge 1$

$$g_i \in H^1(\Gamma_i),$$

$$g_i \in H^{k+1,2}_{\hat{\beta}_i}(\Gamma_i),\ \hat{\beta}_{i,j} \in (\beta_{i+j-1}+1/2, 1),\ j = 1,2$$

$$\|g_i\|_{H^1(\Gamma_i)} \le C\|u\|_{H^{2,2}_{\beta}(\Omega)}$$

$$\|g_i\|_{H^{k+1,2}_{\hat\beta_i}(\Gamma_i)} \le Cd^k \|u\|_{H^{k+2,2}_{\beta}(\Omega)}$$

(iii) If $u \in \mathcal{B}^2_\beta(\Omega)$ and $1/2 < \beta_i, \beta_{i+1} < 1$, then $g_i \in \mathcal{B}^1_{\hat\beta_i}(\Gamma_i)$, $\hat\beta_{i,j} \in (\beta_{i+j-1} - 1/2, 1/2)$, $j = 1,2$. If $0 < \beta_i, \beta_{i+1} < 1/2$ then $g_i \in \mathcal{B}^2_{\hat\beta_i}(\Gamma_i)$, $\hat\beta_{i,j} \in (\beta_{i+j-1} + 1/2, 1)$.

Proof. Without any loss of generality we can assume that $\Gamma_i = \Gamma_1$ and

$\Gamma_1 = \{x_1, x_2 | x_1 \in I,\ x_2 = 0\}$, $A_1 = (-1,0)$, $A_2 = (1,0)$, $\beta = (\beta_1, \beta_2)$.

Let $k \ge 0$ and $v_k = \dfrac{\partial^k u}{\partial x_1^k} \Phi_{k+\beta}$. Then for $k \ge 2$

$$\begin{aligned}
\|v_k\|_{H^2(\Omega)} &\le C[\|(D^2 \frac{\partial^k u}{\partial x_1^k})\Phi_{k+\beta}\|_{L_2(\Omega)} \\
&\quad + k\|(D^1 \frac{\partial^k u}{\partial x_1^k})\Phi_{k+\beta-1}\|_{L_2(\Omega)} \\
&\quad + k^2\|(\frac{\partial^k u}{\partial x_1^k})\Phi_{k+\beta-2}\|_{L_2(\Omega)}] \\
&\le Ck^2 \|u\|_{H^{k+2,2}_\beta(\Omega)} .
\end{aligned}$$

Using Lemma 2.2 we get for $k = 1$

$$|v_1|_{H^2(\Omega)} \le C\|u\|_{H^{3,2}_\beta(\Omega)} .$$

Because of Lemma 2.1 $u \in C^0(\bar\Omega)$ and hence $v_0(A_i) = 0$, $i = 1,2$. Hence using Lemma 2.2 we get

$$|v_0|_{H^2(\Omega)} \le C\|u\|_{H^{2,2}_\beta(\Omega)},$$

and hence for all $k \ge 0$

(4.3) $$\|v_k\|_{H^2(\Omega)} \le C(k+1)^2\|u\|_{H^{k+2,2}_\beta(\Omega)}$$

where C is independent of k. Therefore by the imbedding theorem

$$v_k \in C^0(\bar{\Omega}),\ k \ge 1.$$

Let us show now that $v_k(A_i) = 0,\ i = 1,2,\ k \ge 1$. Assume on the contrary that $v_k^2(A_1) > 0$. Then because $v_k \in C^0(\bar{\Omega})$ we have

$$v_k^2(x) > \varepsilon > 0 \quad \text{for} \quad |x-A_1| < \delta,\ \delta > 0.$$

Hence for $k \ge 2$

$$\infty > \int_\Omega \Phi^2_{k+\beta-2}\left(\frac{\partial^k u}{\partial x_1^k}\right)^2 dx = \int_\Omega \Phi^2_{-2} v_k^2 dx$$

$$\ge \varepsilon^2 \int_{\Omega'_\delta} \Phi^2_{-2} dx = \infty$$

where

$$\Omega'_\delta = \Omega \cap \{x \mid |x-A_1| < \delta\}$$

and we have the desired contradiction. For $k = 1$ we use Lemma 2.2 and get

$$\infty > \int_\Omega \left(\frac{\partial u}{\partial x_1}\right)^2 \Phi^2_{\beta-1} dx > \varepsilon^2 \int_{\Omega'_\delta} \Phi^2_{-2} dx = \infty.$$

If $u \in \mathcal{B}^2_\beta(\Omega)$ then we get from (4.3) for $k > 0$

$$\|v_k\|_{H^2(\Omega)} \le C d_1^k k!.$$

We have $g^{(k)}(x_1) = \frac{\partial^k u}{\partial x_1^k}|_{\Gamma_1}$, $k \geq 0$. Then $g^{(k)}(x_1) = \Phi^{-1}_{k+\beta}(x)v_k(x)|_{\Gamma_1} = \Phi^{-1}_{k+\beta}(x_1)v_k(x_1)$ where we wrote $\Phi^{-1}_{k+\beta}(x_1)$ and $v_k(x_1)$ instead of $\Phi^{-1}_{k+\beta}(x_1,0)$ and $v_k(x_1,0)$. Assume first that $\frac{1}{2} < \beta_1, \beta_2 < 1$. Let $d_0 = \{\min_{j=3,\dots M} \text{dist}\,(A_j,\Gamma_1)\}^{M-2}$. Then we have for $X \in \Gamma_1$, $\hat{\Phi}(x_1) \leq \Phi(x_1)d_0^{-1}$ and hence for $j = 1,2,\dots,k+1$

$$\int_{\Gamma_1} \hat{\Phi}^2_{j-1+\hat{\beta}_1} |g^{(j)}(x_1)|^2 dx_1$$

$$\leq Cj^2\{\int_{-1}^{1} \hat{\Phi}^2_{j-1+\hat{\beta}}[|v'_{j-1}|^2 \Phi^{-2}_{\beta+j-1}$$

$$+ \Phi^{-2}_{j+\beta}|v_{j-1}|^2]dx_1\} \leq Cd_0^{-2j}j^2\{\int_{-1}^{1}[|v'_{j-1}|^2\hat{\Phi}^2_{\hat{\beta}_1-\tilde{\beta}}$$

$$+ |v_{j-1}|^2\hat{\Phi}^2_{-1+\hat{\beta}_1-\tilde{\beta}}]dx_1\}.$$

Using Lemma 2.5 the fact that $j = 1,\dots,k+1$, $v_{j-1}(A_i) = 0$, $i = 1,2$ and that $\tilde{\beta}-\hat{\beta}_1+1 > 1/2$ we get for some $d_1 < 1$

$$\int_{\Gamma_1} \hat{\Phi}^2_{j-1+\hat{\beta}_1} |g^{(j)}(x_1)|^2 dx_1 \leq Cd_1^{-2j} \int_{-1}^{1} |v'_{j-1}|^2 \hat{\Phi}^2_{\hat{\beta}_1-\tilde{\beta}} dx_1.$$

By (4.3) and the imbedding theorem we have for $1 < p < \infty$ and $j = 1,\dots,k+1$

$$\|v'_{j-1}\|_{L_p(I)} \leq C(p)\|v_{j-1}\|_{H^2(\Omega)} \leq Cj^2\|u\|_{H^{j+1,2}_\beta(\Omega)}.$$

Hence for $j = 1,\dots,k+1$, because $\hat{\beta}_1-\tilde{\beta} > -1/2$ we get

$$\int_{-1}^{1} \hat{\Phi}^2_{j-1+\hat{\beta}_1} |g^{(j)}(x_1)|^2 dx \le C d_1^{-2k} \Big(\int_{-1}^{1} (\Phi^2_{\hat{\beta}_1-\tilde{\beta}})^p dx \Big)^{1/p} \| v'_{j-1} \|^2_{L_{2q}(I)}$$

$$\le C d_1^{-2k} \| v_{j-1} \|^2_{H^2(\Omega)} \le C d_2^{-2k} \| u \|^2_{H^{k+2,2}_\beta(\Omega)} .$$

Because by Lemma 2.1

$$\| u \|_{C^0(\bar{\Omega})} \le C \| u \|_{H^{2,2}_\beta(\Omega)}$$

we get

$$\| g \|_{L_2(\Gamma_1)} \le C \| u \|_{H^{2,2}_\beta(\Omega)} .$$

Hence we have proven (i) and (iii) for $1/2 < \beta_i, \beta_{i+1} < 1$ and $k \ge 0$. Assume now that $0 < \beta_1, \beta_2 < 1/2$. We will proceed analogously as before. For $j \ge 2$ we have

$$\int_{\Gamma_1} \hat{\Phi}^2_{j-2+\hat{\beta}_1} |g^{(j)}(x_1)|^2 dx_1$$

$$\le C j^2 \int_{-1}^{1} \hat{\Phi}^2_{j-2+\hat{\beta}_1} [|v'_{j-1}|^2 \Phi^{-2}_{\beta+j-1} + \Phi^2_{j+\beta} |v_{j-1}|^2] dx_1$$

$$\le C d_1^{-2j} \int_{-1}^{1} \hat{\Phi}^2_{-1+\hat{\beta}_1-\tilde{\beta}} (v'_{j-1}(x_1))^2 dx_1$$

when we once used Lemma 2.5 and the fact that $-1+\hat{\beta}_1-\tilde{\beta} > -1/2$. Hence using (4.3) and realizing that $-1+\hat{\beta}_1-\tilde{\beta} > -1/2$ we get analogously as before for $j = 2,\dots,k+1$

$$\int_{\Gamma_1} \hat{\Phi}^2_{j-2+\hat{\beta}_1} |g^{(j)}(x_1)|^2 dx_1 \le C d_2^{2k} \| u \|^2_{H_2^{k+2,2}(\Omega)} .$$

Let us prove now that

$$\|g\|_{H^1(\Gamma_i)} \le C\|u\|_{H_\beta^{k+2,2}(\Omega)}, \quad k \ge 1.$$

We have $v_0(A_1) = v_0(A_2) = 0$ and hence

$$\int_{-1}^{1} g'^2 dx \le C d_0^{-2} \int_{-1}^{1} [\hat{\Phi}_\beta^{-2} |v_0'|^2 + |v_0|^2 \hat{\Phi}_{\tilde{\beta}+1}^{-2}] dx \le \tilde{C} d_0^{-2} \int_{-1}^{1} \hat{\Phi}_\beta^{-2} |v_0'|^2 dx$$

where we have once more used Lemma 2.5. Because $0 < \beta < 1/2$ and

$$\|v_0'\|_{L_p(I)} \le C(p)\|v_0\|_{H^2(\Omega)} \le C\|u\|_{H_\beta^{2,2}(\Omega)}$$

we proceed as before and (ii) and (iii) follow easily.

Remark 4.2. It was essential in the proof of Theorem 4.1 that $\hat{\beta}_{i,j} \in (\beta_{i+j-1}-1/2, 1)$ respectively, $\hat{\beta}_{i,j} \in (\beta_{i+j-1}+1/2, 1)$, i.e., of the open interval. The proof does not hold for the closed interval. It was assumed in Lemma 4.9 that the edge Γ_0 of the domain was straight. Let us assume now that $\Gamma_i = m(\bar{I})$ where $m = (\varphi, \psi)$ are analytic functions on I as given in section 2. Then we have

Lemma 4.9. Let the edge Γ_i of the domain be analytic. Then the part (iii) of Theorem 4.1 holds.

Proof. By Lemma 2.4, $u \in \mathfrak{C}_\beta^2(\Omega)$. Let $M(\xi) = (\varphi(\xi), \psi(\xi))$, $\xi \in I$ be the mapping of I onto Γ_1. Then we define

$$M_1(\xi,\eta) = \varphi(\xi) - \eta\psi'(\xi), \quad M_2(\xi,\eta) = \psi(\xi) + \eta\varphi'(\xi).$$

Then the mapping $M(\xi,\eta) = (M_1(\xi,\eta), M_2(\xi,\eta))$ is analytic on $I_\delta = \{\xi,\eta \,|\, -1-\delta < \xi < 1+\delta, \ |\eta| < \delta\}$, $\delta > 0$, $|J| < \alpha$, $|J^{-1}| < \alpha$ on I_δ (where J is the Jacobian of the mapping) and maps I_δ onto the (open) neighborhood S^* of Γ_i. Denoting $\Omega^* = \Omega \cap S^*$,

$T = M^{-1}(\Omega^*)$, we see that $v(x) = u(M^{-1}(x))$ is defined on T, and $v \in \mathfrak{C}^2_\beta(T)$ by using Lemma 4.7. Hence $v \in \mathfrak{B}^2_{\beta+\varepsilon}(T)$, $\varepsilon > 0$ arbitrary, by Lemma 2.3. Hence for $1/2 < \beta_i, \beta_{i+1} < 1$ we get by (iii) of Theorem 4.1

$g_i(\xi) = v(\xi,0) \in \mathfrak{B}^1_{\hat{\beta}_i}(I_i)$, $\hat{\beta}_{i,j} \in (\beta_{i+j-1}+\varepsilon-1/2,1/2)$, $j = 1,2$.

Because $\varepsilon > 0$ arbitrary $\hat{\beta}_{i,j} \in (\beta_{i+j-1}-1/2,1/2)$. Analogously for $0 < \beta_i, \beta_{i+1} < 1/2$, $g_i(\xi) \in \mathfrak{B}^2_{\hat{\beta}_i}(I)$, $\hat{\beta}_{i,j} \in (\beta_{i+j-1}-1/2,1)$.

<u>Lemma 4.10</u>. Let $g_1 \in \mathfrak{B}^1_{\hat{\beta}}(I)$, $0 < \hat{\beta}_1 < 1/2$, $0 < \hat{\beta}_2 < 1$, $g_2 \in \mathfrak{B}^2_{\hat{\beta}}(I)$, $1/2 < \hat{\beta}_1 < 1$, $0 < \hat{\beta}_2 < 1$. Let $S = \{r,\theta \mid 0 < \theta < 2\pi,\ 0 < r < 1\}$ where (r,θ) are polar coordinates with respect to $(-1,0)$ and $\Phi(r) = r$. Define

$$U_i(r,\theta) = g_i(-1+r)$$

$$V_i(r,\theta) = \theta[g_i(-1+r)-g_i(-1)]$$

(by Lemma 3.1, 3.2, $g_i \in C^0(\bar{I})$, $i = 1,2$, and hence $g_i(-1)$ is well defined). Then

$$U_1, V_1 \in \mathfrak{B}^2_\beta(S),\ \beta = \hat{\beta}_1+1/2$$

$$U_2, V_2 \in \mathfrak{B}^2_\beta(S),\ \beta = \hat{\beta}_1-1/2.$$

<u>Proof</u>. Assume first that $0 < \hat{\beta}_1 < 1/2$ and $g_1 \in \mathfrak{B}^1_{\hat{\beta}}(I)$. Set $\beta = \hat{\beta}_1+1/2$ and $U_1 = g_1(-1+r)$. Then for $k \geq 2$

$$\int_S \left(\frac{\partial^k U_1}{\partial r^k}\right)^2 (r^{k-2+\beta})^2 r\,dr\,d\theta$$

$$\le Cd^{2k} \| g_1^{(k)} \hat{\Phi}_{k-1+\hat{\beta}} \|^2_{L_2(I)}$$

$$\le Cd_1^{2k} (k!)^2 .$$

Hence by Theorem 1.1 of [4] we have for $k \ge 2$, $|\alpha| = k$

$$\| |D^{\alpha} U_1| \Phi_{\beta+k-2} \|_{L_2(S)} \le Cd_2^k k! .$$

Further

$$\|U_1\|_{H^1(S)} \le C \| g_1^{(1)} \cdot \Phi_{1/2} \|_{L_2(I)}$$

$$\le C \| g_1^{(1)} \hat{\Phi}_{\hat{\beta}_1} \|_{L_2(I)} .$$

Hence $U_1 \in \mathcal{B}^2_{\beta}(S)$. Let now $1/2 < \hat{\beta}_1 < 1$. Set $\beta = \hat{\beta}_1 - 1/2$. As before we have for $k \ge 2$

$$\int_S \left(\frac{\partial^k U_2}{\partial r^k}\right)^2 (r^{k-2+\beta})^2 r dr d\theta \le Cd_1^{2k}(k!)^2$$

and we get $\|U_2\|_{H^1(S)} < \infty$. Hence $U_i \in \mathcal{B}^2_{\beta}(S)$. Let us consider now the function $V_1(r,\theta)$. Then as before

$$\int_S \left(\frac{\partial^k V_1}{\partial r^k}\right)^2 (r^{k-2+\beta})^2 r dr d\theta \le Cd_1^{2k}(k!)^2 .$$

Further, using Lemma 2.5 and $k \ge 2$ we get

$$\int_S \left(\frac{\partial^k V_1}{\partial r^{k-1}\partial\theta}\right)^2 r^{-2}(r^{k-2+\beta})^2 r dr d\theta = \int_S \left(\frac{\partial^k g_1}{\partial r^{k-1}}\right)^2 r^{-2}(r^{k-2+\beta})^2 r dr d\theta$$

$$\le Cd_1^{2k} \| (g_1^{(k-1)} \hat{\Phi}_{k-2+\hat{\beta}_1} \|^2_{L_2(I)}$$

$$\le Cd^{2k}[\|(g_1^{(k-1)}-g_1^{(k-1)}(0))\hat{\Phi}_{k-2+\hat{\beta}_1}\|^2_{L_2(I)}$$

$$+ (g_1^{(k-1)}(0))^2\|\hat{\Phi}_{k-2+\hat{\beta}_1}\|^2_{L_2(I)}]$$

$$\le Cd_2^{2k}[(g_1^{(k-1)}(0))^2 + \|g_1^{(k)}\hat{\Phi}_{k-1+\hat{\beta}_1}\|^2_{L_2(I)}]$$

$$\le Cd_3^{2k}(k!)^2.$$

In the last inequality we used the fact that

$$|g^{(k-1)}(0)| \le Cd_4^k(k!)$$

and realizing that $\dfrac{\partial^k V_1}{\partial r^{k-j} d\theta^j} = 0$ for $j \ge 2$ we have for $k \ge 2$

$$\||D^\alpha V_1|\Phi_{\beta+k-2}\|_{L_2(S)} \le Cd^k k!.$$

Further for $0 < \hat{\beta}_1 < 1/2$ and $I^* = (-1,0)$

$$\|V_1\|^2_{H^1(S)} \le C[\|g_1^{(1)}\hat{\Phi}_{1/2}\|^2_{L(I^*)} + \|(g_1(x)-g_1(-1))\hat{\Phi}_{-1/2}\|^2_{L_2(I^*)}]$$

$$\le C[\|g_1^{(1)}\hat{\Phi}_{\hat{\beta}}\|^2_{L(I^*)} + \|(g_1(x)-g_1(-1))\hat{\Phi}_{-1+\hat{\beta}_1}\|_{L_2(I^*)}]$$

$$\le C[\|g_1^{(1)}\hat{\Phi}_{\hat{\beta}_1}\|_{L(I^*)} + \|(g_1^{(1)}\hat{\Phi}_{\hat{\beta}_1}\|_{L_2(I)}]$$

$$\le C\|g_1\|_{H^{1,1}_\beta(I)}.$$

In the last inequality we have used once more lemma 2.5 and the fact that $\hat{\beta}_1 < 1/2$. Quite analogously we prove that $V_2 \in \mathcal{B}^2_\beta(S)$.

__Lemma 4.11.__ Let $g \in \mathcal{B}^{1}_{\hat{\beta}}(I)$, $0 < \hat{\beta} < 1/2$, $g(\pm 1) = 0$. then for $0 < \gamma < 1/2$, $v = g\hat{\Phi}_{-\gamma} \in \mathcal{B}^{1}_{\hat{\beta}+\gamma}(I)$.

__Proof.__ For $k \geq 1$

$$\int_{-1}^{1} ((v^{(k)})^2 \hat{\Phi}^2_{k-1+\hat{\beta}+\gamma})dx$$

$$\leq \int_{-1}^{1} \left[\sum_{\ell=0}^{k} \binom{k}{\ell} g^{(\ell)} (\hat{\Phi}_{-\gamma})^{(k-\ell)}\right]^2 \hat{\Phi}^2_{k-1+\hat{\beta}+\gamma} dx$$

$$\leq Cd^{2k} \sum_{\ell=0}^{k} \int_{-1}^{1} (g^{(\ell)})^2 ((k-\ell)!)^2 \hat{\Phi}^2_{-\gamma-(k-\ell)} \hat{\Phi}^2_{k-1+\hat{\beta}+\gamma} dx$$

$$\leq Cd^{2k} \left[\sum_{\ell=1}^{k} \int_{-1}^{1} (g^{(\ell)})^2 \hat{\Phi}^2_{\hat{\beta}+\ell-1} ((k-\ell)!)^2 dx + \int_{-1}^{1} (g)^2 \hat{\Phi}^2_{\hat{\beta}-1} (k!)^2 dx\right]$$

$$\leq Cd^{2k} \left[\sum_{\ell=1}^{k} \int_{-1}^{1} (g^{(\ell)})^2 \hat{\Phi}_{\hat{\beta}+\ell-1} ((k-\ell)!)^2 dx + \int_{-1}^{1} (g')^2 \hat{\Phi}^2_{\hat{\beta}} (k!)^2 dx\right] \leq Cd_1^{2k} (k!)^2$$

when we have used Lemma 2.5 in the above inequality. Further

$$\int_{-1}^{1} v^2 dx = \int_{-1}^{1} g^2 \hat{\Phi}^2_{-\gamma} dx \leq C\|g\|^2_{H^{1,1}_{\hat{\beta}}(I)}$$

by Lemma 4.1.

Lemma 4.12. Let $g \in \mathcal{B}^2_{\hat\beta}(I)$, $g(\pm 1) = 0$, $1/2 < \hat\beta < 1$, $0 < \gamma < 1/2$, $v = g\hat\Phi_{-\gamma}$. Then for $\hat\beta+\gamma > 1$, $v \in \mathcal{B}^1_{\hat\beta+\gamma-1}(I)$ and for $\hat\beta+\gamma < 1$, $v \in \mathcal{B}^2_{\hat\beta+\gamma}(I)$.

Proof. (a) Assume first that $\hat\beta+\gamma > 1$. Then for $k \geq 2$

$$\int_{-1}^{1} (v^{(k)})^2 \hat\Phi^2_{k+\hat\beta+\gamma-2}\,dx$$

$$\leq Cd^{2k}\left[\sum_{\ell=2}^{k} \int_{-1}^{1} (g^{(\ell)})^2 \hat\Phi_{-\gamma-(k-\ell)+k+\hat\beta+\gamma-2}((k-\ell)!)^2 dx\right.$$

$$\left. + (k!)^2 \int_{-1}^{1} g^2 \hat\Phi^2_{\hat\beta-2}\,dx + ((k-1)!)^2 \int_{-1}^{1} g'^2 \hat\Phi^2_{\hat\beta-1}\,dx\right]$$

$$\leq Cd^{2k}\left[\sum_{\ell=2}^{k} \int_{-1}^{1} (g^{(\ell)})^2 \hat\Phi^2_{\hat\beta+\ell-2}((k-\ell)!)dx\right.$$

$$\left. + (k!)^2 \int_{-1}^{1} g'^2 \hat\Phi^2_{\hat\beta-1}\,dx\right].$$

In the last inequality Lemma 2.5 has been used. Because by the imbedding theorem $|g'(0)| \leq C\|g\|_{H^{2,2}_{\hat\beta}(I)}$ using Lemma 2.5 once more rendering that $\hat\beta-1 > -1/2$ we get

$$\int_{-1}^{1} g'^2 \hat\Phi^2_{\hat\beta-1}\,dx \leq C\left[\int_{-1}^{1} g''^2 \hat\Phi^2_{\hat\beta}dx + |g'(0)|^2\right] \leq C\|g\|^2_{H^{2,2}_{\hat\beta}(I)} < \infty.$$

Hence

$$\int_{-1}^{1} (v^{(k)})^2 \hat\Phi_{k+\hat\beta+\gamma-2}\,dx \leq Cd_1^{2k}(k!)^2.$$

Further as before

$$\int_{-1}^{1} v'^2 \hat{\Phi}^2_{\hat{\beta}+\gamma-1}\,dx \le C\int_{-1}^{1} g'^2\hat{\Phi}^2_{\hat{\beta}-1}\,dx \le C\|g\|^2_{H^{2,2}_{\hat{\beta}}(I)} < \infty.$$

Because $g \in C^0(\bar{I})$, $v \in L_2(I)$.

(b) Assume now that $\hat{\beta}+\gamma < 1$. Then for $k \ge 2$ we get exactly as before that

$$\int_{-1}^{1} (v^{(k)})^2\hat{\Phi}_{k+\hat{\beta}+\gamma-2}\,dx \le Cd_1^{2k}(k!)^2.$$

Further

$$\int_{-1}^{1} v'^2dx \le C\left[\int_{-1}^{1} g^2\hat{\Phi}^2_{-\gamma-1}dx + \int_{-1}^{1} g'^2\hat{\Phi}^2_{-\gamma}dx\right]$$

$$\le C\left[\int_{-1}^{1} g''^2\hat{\Phi}^2_{-\gamma+1}dx + |g'(0)|^2\right].$$

Because $-\gamma+1 > \hat{\beta}$ by our assumption we see that

$$\int_{-1}^{1} v'^2dx \le C\|g\|^2_{H^{2,2}_{\hat{\beta}}(I)}.$$

Using Lemma 4.2 we get also $\|v\|_{L_2(I)} \le C\|g\|_{H^{2,2}_{\beta}(I)}$.

<u>Lemma 4.13</u>. Let Ω be a curvilinear polygon with the vertices A_i, $i = 1,\dots,M$. Let $u \in \mathcal{B}^2_\beta(\Omega)$ and w be such that

$$|D^\alpha w| \le C\Phi_{-|\alpha|+\gamma}|\alpha|!d^{|\alpha|},$$

$$\gamma = (\gamma_1,\dots,\gamma_M),\ |\alpha| \ge 0,\ \beta_i-\gamma_i > 0,\ \gamma_i > 0.$$

Then $v = wu \in \mathcal{B}^2_{\bar{\beta}}(\Omega)$ where $\bar{\beta}_i = \beta_i-\gamma_i$.

<u>Proof</u>. For $k \geq 2$, $|\alpha| = k$,

$$\int_\Omega |D^{|\alpha|}v|^2 \Phi^2_{|\alpha|-2+\bar\beta}dx \leq Cd^{2k}\left[\sum_{\ell=0}^{k} \int_\Omega |D^{k-\ell}u||D^\ell w|\right]^2 \Phi^2_{k-2+\bar\beta}dx$$

$$\leq Cd_1^{2k}\sum_{\ell=0}^{k}((\ell+1)!)^2\int_\Omega |D^{k-\ell}u|^2\Phi^2_{k-2-\ell+\beta}dx$$

$$\leq Cd_1^{2k}\sum_{\ell=0}^{k}((\ell+1)!)^2((k-1+\ell)!)^2 < Cd_2^{2k-2}((k-2)!)^2 .$$

Further

$$\int_\Omega |D^1 v|^2 dx \leq C\left[\int_\Omega |D^1 u|^2|w|^2 dx + \int_\Omega |u|^2|D^1 w|^2 dx\right] < \infty$$

because by lemma 2.1 $u \in C^0(\bar\Omega)$.

It is very easy to prove

<u>Lemma 4.14</u>. Let $g \in \mathcal{B}^0_{\hat\beta}(I)$, $0 < \hat\beta < 1/2$. Then $v = g\hat\Phi \in \mathcal{B}^1_{\hat\beta}(I)$ and $v(\pm 1) = 0$. Let $g \in \mathcal{B}^1_{\hat\beta}(I)$, $1/2 < \hat\beta < 1$ then $v = g\hat\Phi \in \mathcal{B}^2_{\hat\beta}(I)$ and $v(\pm 1) = 0$.

<u>Proof</u>. The statement that $v \in \mathcal{B}^1_{\hat\beta}(I)$ can be directly verified. By Lemma 4.1 v is continuous on $\bar I$. If $v(-1) \neq 0$ then $v^2(x) > \varepsilon > 0$ for all $|x+1| < \delta$. Hence $g^2 = (v\hat\Phi^{-1})^2 \geq \varepsilon\hat\Phi^2_{-1}$ which would contradict with the assumption that $g \in \mathcal{B}^0_{\hat\beta}(I)$, $0 < \hat\beta < 1/2$. The proof of the second part of the lemma is analogous.

<u>Lemma 4.15</u>. Let $u \in \mathcal{B}^2_\beta(\Omega)$, $0 < \beta < 1$ and $u = 0$ at A_i. Then $u\Phi^{-1} \in \mathcal{B}^1_\beta(\Omega)$. The proof follows easily using Lemma 2.2.

<u>Theorem 4.2</u>. Let Ω be a straight polygon with the edges Γ_i,

$i = 1,\dots,M$, and let $g \in \mathcal{B}^1_\beta(\Gamma_1)$, $0 < \bar\beta_i < 1/2$, $\beta_i = \bar\beta_i + 1/2$, $i = 1,2$ (respectively $g \in \mathcal{B}^2_\beta(\Gamma_1)$, $1/2 < \bar\beta_i < 1$, $\beta_i = \bar\beta_i - 1/2$, $i = 1,2$) and $g(A_i) = 0$, $i = 1,2$. Then there is u such that

(i) $u \in \mathcal{B}^2_\beta(\Omega)$, with $0 < \beta_j < 1$, $j = 3,\dots,M$, arbitrary.

(ii) $u|_{\Gamma_1} = g$ and $u|_{\Gamma_j} = 0$ for $j = 2,\dots,M$.

<u>Proof</u>. Let $\varphi = \prod_{i=3}^{M} |x-A_i|^2$, $x \in \Omega$. Denote $\tilde g = g/\varphi$. Then obviously $\tilde g \in \mathcal{B}^1_\beta(\Gamma_1)$ (respectively $\tilde g \in \mathcal{B}^2_\beta(\Gamma_1)$). Select now $0 < \gamma_i < 1/2$ such that $0 < \bar\beta+\gamma_i < 1/2$ (respectively $0 < \bar\beta+\gamma_i-1 < 1/2$). Denote $\hat g = \tilde g \prod_{i=1}^{2} |x-A_i|^{-\gamma_i} = \tilde g \Phi_{-\gamma}$ where $\gamma = (\gamma_1,\gamma_2,0,\dots,0)$ By Lemma 4.1 and 4.2 $\hat g(A_i) = 0$, $i = 1,2$. Using Lemma 4.11 (and 4.12) we see that $\hat g \in \mathcal{B}^1_{\beta+\gamma}(I)$ (respectively $\hat g \in \mathcal{B}^1_{\beta+\gamma-1}(I)$).

Let $U \in H^1(\Omega)$, $\Delta U = 0$ and $U = \hat g$ on Γ_1 and $U = 0$ on Γ_j, $j = 2,\dots,M$. Function U exists and is uniquely determined. To see it let $\varphi(x)$, $x \in \Gamma_1$, $\varphi \in C^\infty(\Gamma_1)$, $\varphi(x) = 1$ for $|x-A_i| < \varepsilon/2$, $i = 1,2$ and $\varphi(x) = 0$ for $|x-A_i| > \varepsilon$, $i = 1,2$ with ε sufficiently small. We define

$$U = U_1+U_2$$

where $\Delta U_i = 0$, $U_i \in H^1(\Omega)$, $i = 1,2$, $U_1|_{\Gamma_1} = \hat g(1-\varphi)$, $U_2|_{\Gamma_1} = \hat g\varphi$ and $U_i = 0$ on Γ_j, $j = 2,\dots,M$. Because $h_1 = \hat g(1-\varphi) \in C^\infty(\Gamma_1)$ and $h_1(x) = 0$ for $|x-A_i| < \varepsilon/2$, U_1 obviously exists.

By Lemma 4.10 there exists $W \in H^1(\Omega)$ such that $W|_{\Gamma_1} = h_2 = \hat g\varphi$, and $W|_{\Gamma_j} = 0$, $j = 2,\dots,M$. Hence U_2 exists too. Function

U has the following properties:

(i) $\Delta U = 0$.

(ii) $U|_{\Gamma_1} = \hat{g}$, $U|_{\Gamma_j} = 0$, $j = 2,\dots,M$.

(iii) $\hat{g}$ is analytic on Γ_1 (not on $\bar{\Gamma}_1$).

(iv) in $\Omega_{i,\delta} = \Omega \cap \{x \mid |x-A_i| < \delta\}$, $i = 1,2$ with δ sufficiently small there is W_i such that $W_i \in \mathcal{B}^2_{\bar{\beta}}(\Omega_{i,\delta})$ where $\bar{\beta} = \hat{\beta}+\gamma+1/2$ (respectively $\bar{\beta} = \hat{\beta}+\gamma-1+1/2$) and $W_i|_{\Gamma_i \cap \bar{\Omega}_{i,\delta}} = \hat{g}$. (This follows from Lemma 4.10.)

By the selection of γ_i we have $\bar{\beta}_i > 1/2$, $i = 1,2$. Using now the same arguments as in the proof of Theorem 2.1 in [4] we conclude that $U \in \mathcal{B}^2_{\bar{\beta}}(\Omega)$ where $\bar{\beta}_i = \hat{\beta}_i + \gamma_i + 1/2$ (respectively $\bar{\beta}_i = \hat{\beta}_i + \gamma_i - 1/2)$, $i = 1,2$, and $1 > \bar{\beta}_j > 1/2$.

By Lemma 4.13 we see that $u = \psi\Phi_\gamma U \in \mathcal{B}^2_\beta(\Omega)$ where $\beta_i = \hat{\beta}_i + 1/2$ (respectively $\beta_i = \hat{\beta}_i - 1/2)$, $i = 1,2$ and $0 < \beta_j < 1$ arbitrary for $j = 3,\dots,M$. In addition $u|_{\Gamma_1} = g$ and $u|_{\Gamma_j} = 0$, $j = 2,\dots,M$.

Let us outline the main idea of the assertion that $U \in \mathcal{B}^2_{\bar{\beta}}(\Omega)$. Let $S_{i,\delta_i} = \{r_i,\theta_i \mid 0 < r_i < \delta_i,\ 0 < \theta_i < r_i\} \cap \Omega$ where (r_i,θ_i) are the polar coordinates with the origin in A_i. We select $\delta_0 < 1$ such that $S_{i,2\delta_i} \cap S_{j,2\delta_j} = \emptyset$ for $i \neq j$. Using Theorems 5.7.1, 5.7.1' and 6.6.1 of [17] we conclude similarly, as in the proof of Theorem 2.1 of [4], that $U \in \mathcal{B}^2_{\bar{\beta}}(\Omega - \bigcup_{i=1}^{M} S_{i,\delta_i/4})$ due to the analyticity of $\hat{g}$ on $\Gamma - \bigcup_{i=1}^{M} S_{i,\delta_i/4}$. Hence we have to prove only that $U \in \mathcal{B}^2_{\beta}(S_{i,\delta_i/4})$.

Let

$$\varphi_0 \in C^{\infty}(\mathbb{P}^+)$$

$$\varphi_0(1) = 1 \quad \text{for} \quad 0 \le r \le 1/2$$

$$\varphi_0(0) = 0 \quad \text{for} \quad x \ge 1$$

$$\varphi_{\delta_i}(r) = \varphi_0(\frac{r}{2\delta_i}) = \varphi(r).$$

Denote $v = \varphi U$. Then v can be understood to be defined on the infinite sector $Q^{(i)}_{w_i} = \{(r_i,\theta_i) | 0 < r_i < \infty,\ 0 < \theta < \omega_i\}$ when extended by zero outside of S_{i,δ_i} and we have $v \in H^1(Q^{(i)}_{w_i})$.

Now we prove that $v \in \mathfrak{B}^2_\beta(S_{i,\delta_i/2})$ as in [4].

<u>Remark 4.3</u>. We have assumed that either $g \in \mathfrak{B}^1_{\hat\beta}(\Gamma_1)$, $0 < \hat\beta < 1/2$ or $g \in \mathfrak{B}^2_{\hat\beta}(\Gamma_1)$, $1/2 < \hat\beta < 1$. Obviously Theorem 4.2 is correct if $g \in \mathfrak{B}^1_{\hat\beta}(\Gamma_1)$ only in the neighborhood of A_1 and $g \in \mathfrak{B}^2_{\hat\beta}(\Gamma_1)$ in the neighborhood of A_2. Theorem 4.1 leads easily to the next theorem.

<u>Theorem 4.3</u>. Let Ω be a straight polygon with the edges Γ_i, $i = 1,\dots,M$ and let

$$g \in \mathfrak{B}^1_{\hat\beta_i}(\Gamma_i),\ \hat\beta_i = (\hat\beta_{i,1},\hat\beta_{i,2}),\ 0 < \hat\beta_{i,1},\hat\beta_{i,2} < 1/2,$$

$$\bar\beta_{i,1} = \hat\beta_{i,1}+1/2,\ \bar\beta_{i,2} = \hat\beta_{i,2}+1/2$$

or

$$g \in \mathfrak{B}^2_{\hat\beta_i}(\Gamma_i),\ \hat\beta_i = (\hat\beta_{i,1},\hat\beta_{i,2}),\ 1/2 < \hat\beta_{i,1},\hat\beta_{i,2} < 1,$$

$$\bar\beta_{i,1} = \hat\beta_{i,1}-1/2,\ \bar\beta_{i,2} = \hat\beta_{i,2}-1/2,\ i \in Q \subset \{1,\dots,M\}.$$

Let further g be continuous on $\gamma = \bigcup_{i\in Q} \bar\Gamma_i$. Then $g \in \mathfrak{B}^{3/2}_{\bar\beta}(\gamma)$

where $\bar{\beta}_i = \max(\bar{\beta}_{i-1,2}, \bar{\beta}_{i,1})$, for $A_i \in \gamma$ (if $i-1 \notin Q$ or $i \notin Q$ then we define $\bar{\beta}_{i-1,2} = 0$ respectively $\bar{\beta}_{i,1} = 0$) and $0 < \bar{\beta}_i < 1$ arbitrary for $A_i \notin \gamma$.

Proof. Because g is continuous on γ we can construct a polynomial P on Ω such that $g-P = 0$ at A_i. Then we can apply Theorem 4.2.

Remark 4.4. It is obvious how the theorem may be modified when $g \in \mathfrak{B}^1_\beta(\Gamma_i)$ respectively $g \in \mathfrak{B}^2_\beta(\Gamma_i)$ in the neighborhood of A_i only. See also Remark 4.3.

Remark 4.5. Theorem 4.1 and Theorem 4.3 are complementary, which is analogous to the theorems of trace and extension in usual Sobolev spaces on smooth domain, namely, if $g \in \mathfrak{B}^1_{\hat{\beta}_i}(\Gamma_i)$, $0 < \hat{\beta}_{i,j} < 1/2$ (respectively $g \in \mathfrak{B}^2_{\hat{\beta}_i}(\Gamma)$, $1/2 < \hat{\beta}_{i,j} < 1$) $j = 1,2$, then we have an extension by function $G \in \mathfrak{B}^2_\beta(\Omega)$, $\beta_i = \hat{\beta}_{i,1}+1/2$, $\beta_{i+1} = \hat{\beta}_{i,2}+1/2$ (respectively $\beta_i = \hat{\beta}_{i,1}-1/2$, $\beta_{i+1} = \hat{\beta}_{i,2}-1/2$), and if $G \in \mathfrak{B}^2_\beta(\Omega)$ then $G|_{\Gamma_i} = g \in \mathfrak{B}^1_{\hat{\beta}_i+\varepsilon}(\Gamma_i)$, $\hat{\beta}_{i,1} = \beta_i-1/2$, $\hat{\beta}_{i,2} = \beta_{i+1}-1/2$ for $1/2 < \beta_i, \beta_{i+1} < 1$ (respectively $g \in \mathfrak{B}^2_{\hat{\beta}_i+\varepsilon}(\Gamma_i)$, $\hat{\beta}_{i,1} = \beta_i+1/2$, $\hat{\beta}_{i,2} = \beta_{i+1}+1/2$ for $0 < \beta_i, \beta_{i+1} < 1/2$), $\varepsilon > 0$ arbitrary.

Theorem 4.4. Let Ω be a straight polygon with the edges Γ_i, $i = 1,\dots,M$, and let $g \in \mathfrak{B}^0_{\hat{\beta}}(\Gamma_1)$, $0 < \hat{\beta}_i < 1/2$, $i = 1,2$, $\beta_i = \hat{\beta}_i+1/2$, $i = 1,2$ (respectively $g \in \mathfrak{B}^1_{\hat{\beta}}(\Gamma_1)$, $1/2 < \hat{\beta}_i < 1$, $\beta_i = \hat{\beta}_i-1/2$, $i = 1,2$). Then there is u such that

(i) $u \in \mathfrak{B}^1_\beta(\Omega)$ with $0 < \beta_j < 1$, $j = 3,\dots,M$ arbitrary.

(ii) $u|_{\Gamma_1} = g$ and $u|_{\Gamma_j} = 0$, $j = 2,\dots,M$.

Proof. By Lemma 4.14, $\tilde{g} = g\hat{\Phi} \in \mathfrak{B}^1_{\hat{\beta}}(\Gamma_1)$ respectively $\mathfrak{B}^2_{\hat{\beta}}(\Gamma_1)$ and $\tilde{g}(A_i) = 0$, $i = 2,3$, and hence by Theorem 4.2 there is $v \in \mathfrak{B}^2_{\beta}(\Omega)$ such that $v = \tilde{g}$ on Γ_1 and $v = 0$ on Γ_j, $j = 2,\dots,M$. By Lemma 4.15 the function $v\Phi^{-1}$ has the desired properties. $\square$

Theorem 4.4 leads immediately to

Theorem 4.5. Let Ω be a straight polygon with the edges Γ_i, $i = 1,\dots,M$ and let

$$g \in \mathfrak{B}^0_{\hat{\beta}_i}(\Gamma_i),\ \hat{\beta}_i = (\hat{\beta}_{i,1},\hat{\beta}_{i,2}),\ 0 < \hat{\beta}_{i,1},\hat{\beta}_{i,2} < 1/2,$$

$$\bar{\beta}_{i,1} = \hat{\beta}_{i,1}+1/2,\ \bar{\beta}_{i,2} = \hat{\beta}_{i,2}+1/2$$

or

$$g \in \mathfrak{B}^1_{\hat{\beta}_i}(\Gamma_i),\ \hat{\beta}_i = (\hat{\beta}_{i,1},\hat{\beta}_{i,2}),\ 1/2 < \hat{\beta}_{i,1},\hat{\beta}_{i,2} < 1,$$

$$\bar{\beta}_{i,1} = \hat{\beta}_{i,1}-1/2,\ \bar{\beta}_{i,2} = \hat{\beta}_{i,2}-1/2,\ i \in Q \subset \{1,\dots,M\}.$$

Let $\gamma = \bigcup_{i\in Q} \bar{\Gamma}_i$. Then $g \in \mathfrak{B}^{1/2}_{\bar{\beta}}(\gamma)$ where $\bar{\beta}_i = \max(\bar{\beta}_{i-1,2},\bar{\beta}_{i,1})$, $A_i \in \gamma$ (if $i-1 \notin Q$ or $i \notin Q$ when we define $\hat{\beta}_{i-1,2} = 0$ respectively $\bar{\beta}_{i,1} = 0$) and $0 < \bar{\beta}_i < 1$ arbitrary for $A_i \notin \gamma$. $\square$

Remark 4.6. It is obvious how Theorem 4.4 has to be modified when $g \in \mathfrak{B}^1_{\hat{\beta}}(\Gamma_i)$ respectively $g \in \mathfrak{B}^2_{\hat{\beta}}(\Gamma_i)$ in the neighborhood of A_i only. See Remark 4.3.

Theorems 4.3 and 4.5 give the characterization of the boundary conditions which guarantees that the solution of an elliptic partial differential equation of second order with analytic coefficients on a domain Ω with piecewise analytic boundary belong to $\mathcal{B}^2_\beta(\Omega)$ or $\mathcal{C}^2_\beta(\Omega)$ (see Theorems 3.2 and 3.3).

In the concrete cases these conditions are usually very easy to check. Let us state a useful lemma which characterizes the space $\mathcal{B}^1_{\hat{\beta}}(I)$ (respectively $\mathcal{B}^2_{\hat{\beta}}(I)$).

Lemma 4.16. Let

$$\Omega_\alpha = \{z = x+iy \mid x \in I,\ |y| \le \alpha\hat{\Phi}(x),\ \alpha > 0\}$$

and $G(z)$ be holomorphic function on Ω_α such that for $\nu = (\nu_1,\nu_2)$

$$|G(z)| \le C\hat{\Phi}_\nu(\mathrm{Re}\ z).$$

Let $g(x) = \mathrm{Re}\ G(z)|_I$ or $\mathrm{Im}\ G(z)|_I$. Then for $\nu_i > -1/2+(j-1)$, $\hat{\beta}_i+\nu_i > 1/2+(j-1)$, $0 < \hat{\beta}_i < 1$, $i = 1,2$, $j = 0,1,2$

$$g(x) \in \mathcal{B}^j_{\hat{\beta}}(I).$$

Proof. By Cauchy formula we have for $k > 0$

$$|g^{(k)}(x)| \le C\hat{\Phi}_\nu(x)(\hat{\Phi}(x))^{-k}k!\alpha^{-k}.$$

Hence

$$\int_{-1}^{1} \Phi^2_{k-1+\hat{\beta}}|g^{(k)}(x)|^2dx \le (Ck!\alpha^{-k})^2\int_{-1}^{1} \Phi^2_{\nu+\hat{\beta}-1}\,dx \le (C_1d^kk!)^2$$

provided that $\nu_i + \hat{\beta}_i > 1/2$. Further for $k = 0$

$$|g(x)| \le C\Phi_\nu(x)$$

and hence for $\nu_i > -1/2$, $g \in H^0(I)$. The lemma is proven for $j = 1$. The proof of the case $j = 0$ is analogous. Let us consider now the case $j = 2$. We see that for $\nu_i + \hat{\beta}_i > 3/2$ and $k \ge 2$

$$\int_{-1}^{1} \Phi^2_{k-2+\hat{\beta}} |g^{(k)}(x)|^2 dx \le (Ck!\alpha^{-k})^2 \int_{-1}^{1} \Phi^2_{\nu-k+k-2+\hat{\beta}} dx \le (C_1 d^k k!)^2 .$$

Further if $\nu_i > 1/2$ then also $g \in H^1(I)$.

Instead of $|G(z)| \le C\hat{\Phi}_\nu(\mathrm{Re}\, z)$ we can assume that $|G(z) - P(z)| \le C\hat{\Phi}_\nu(\mathrm{Re}\, z)$ where $P(z)$ is a polynomial.

Lemma 4.16 is very useful in practice. For example if g is analytic on $\bar{\Gamma}_i$ then $g(x)$ can be extended into some neighborhood of $\bar{\Gamma}_i$ and therefore $g \in \mathcal{B}^1_\beta(I)$. Lemma 4.16 characterizes very well the structure of the spaces $\mathcal{B}^1_\beta(I)$ (respectively $\mathcal{B}^2_\beta(I)$).

<u>Lemma 4.17</u>. Let $g \in \mathcal{B}^1_{\hat{\beta}}(I)$, $0 < \hat{\beta}_i < 1/2$. Then there exists $\alpha > 0$ such that g can be analytically extended onto Ω_α and

$$|G(z) - g(-1)\frac{(1-x)}{2} - g(1)\frac{(x+1)}{2}| \le C\Phi_{1/2-\hat{\beta}}(\mathrm{Re}\, x)$$

($g \in C^0(\bar{I})$ by Lemma 3.1).

<u>Proof</u>. Since $g \in \mathcal{B}^1_{\hat{\beta}}(I)$ we have by Lemma 4.3 for $k \ge 1$

$$|g^{(k)}(x)| \le C\left[\Phi^2_{k-1/2+\hat{\beta}}(x)\right]^{-1} d^k k! .$$

Hence the series

$$g'(x) = \sum_{k=0}^{\infty} g^{(k+1)}(x_0)(x-x_0)^k \frac{1}{k!}, \quad x_0 \in I$$

is absolutely convergent for $|x-x_0| \le \frac{\Phi(x_0)}{d}\frac{1}{2}$ and hence also

$$G'(z) = \sum_{k=0}^{\infty} g^{(k+1)}(x_0)(z-x_0)^k \frac{1}{k!}$$

converges for $|z-x_0| \le \frac{\Phi(x_0)}{d}\frac{1}{2}$ and $|G'(z)| \le C\hat{\Phi}^{-1}_{\beta+1/2}(x_0)$, $x_0 =$ Re(z), and C is independent of x_0, which yields the lemma.

So far we have assumed that Ω is a straight polygon. We did not exclude the case that the internal angle is 2π, i.e., we did not exclude the slip domain. Let us now consider the curvilinear polygon and assume that it is a Lipschitzian domain. let us prove first

<u>Lemma 4.18</u>. Let $\Omega = \{x_1,x_2 | -1 < x_1 < 1,\ 0 < x_2 < h(x_1),\ h(x_1) > \alpha(x_1+1),\ h(-1) = 0,\ \alpha > 0\}$. Assume that $\psi(x_1,x_2)$ is an analytic function on $S = \{x_1,x_2 | (x_1+1)^2 + x_2^2 \le 4\}$ such that

(i) $\psi(x_1,h(x_1)) = 0$.

(ii) $\frac{\partial\psi}{\partial x_1}(x_1,0) > \alpha > 0,\ -1 \le x_1 \le 1$.

Define

$$\Gamma_1 = \{x_1,x_2 | -1 < x_1 < 1,\ x_2 = 0\}$$

$$\Gamma_2 = \{x_1,x_2 | -1 < x_1 < 1,\ x_2 = h(x_1)\}$$

and let $T = \Omega \cap S_1$ where $S_1 = \{r,\theta | 0 < \theta < 2\pi,\ 0 < r < 1\}$ where (r,θ) are polar coordinates with respect to $(-1,0)$ and $T^* = S_1 - T$. Let $g_1 \in \mathcal{B}^1_{\hat{\beta}}(\Gamma_1)$, $0 < \hat{\beta}_i < 1/2$, $\hat{\beta}_1 = \hat{\beta}_2$ (respectively $g_2 = \mathcal{B}^2_{\hat{\beta}}(\Gamma_1)$, $1/2 < \hat{\beta}_i < 1$, $\hat{\beta}_1 = \hat{\beta}_2$), $g_i(-1) = 0$, $i = 1,2$ and $\Phi = r$.

Then there exists

$$V_1 \in \mathcal{B}^2_{\bar\beta}(T),\ V_1^* \in \mathcal{B}^2_{\bar\beta}(T^*),\ \bar\beta = \hat\beta + 1/2$$

(respectively $V_2 \in \mathcal{B}^2_{\bar\beta}(T),\ V_2^* \in \mathcal{B}^2_{\bar\beta}(T^*),\ \bar\beta = \hat\beta - 1/2$)

such that $V_i = g_i$ and $V_i^* = g_i$ on $\Gamma_1 \cap \bar T$ and $V_i, V_i^* = 0$ on $\Gamma_2 \cap \bar T$.

<u>Proof</u>. Let $\varphi(r,\theta) = \psi(r,\theta)\frac{1}{r}$. Then $\varphi(r,0) = \varphi(x_1)$ is analytic on $\bar\Gamma_1$ and $\varphi(x_1) > \bar\alpha > 0$, hence $\varphi^{-1}(x_1)$ is analytic on $\bar\Gamma_1$ too. In addition $\varphi = 0$ on Γ_2. Further $|D^\alpha\varphi(x_1,x_2)| \le C|\alpha|!\Phi^{-|\alpha|}d^{|\alpha|}$ by Cauchy theorem of the theory of two complex variables. Define $\tilde g_1 = g_1\varphi^{-1}(x_1)$. Then $\tilde g_1 \in \mathcal{B}^1_{\hat\beta}(\Gamma_1)$ and by Lemma 4.11 there exists U_1 on S_1 such that $U_1 \in \mathcal{B}^2_{\bar\beta}(S_1)$, $\bar\beta = \hat\beta + 1/2$ and $U_1|_{\Gamma_1} = \tilde g_1$. Define now $V_1 = U_1\varphi$. Using Lemma 4.13 we conclude that $V_1 \in \mathcal{B}^2_{\bar\beta}(T)$ (respectively $\mathcal{B}^2_{\bar\beta}(T^*)$), $V_1|_{\Gamma_1} = g_1$ and $V_1|_{\Gamma_2} = 0$. The proof that V_2 has desired properties is quite analogous.

<u>Lemma 4.19</u>. Let $\Omega = \{x_1,x_2 | -1 < x_1 < 1,\ h_1(x_1) < x_2 < h_2(x_1),\ h_1(x_1) < -\alpha(x_1+1),\ h_2(x_1) > \alpha(x_1+1),\ \alpha > 0,\ h_i(-1) = 0,\ h_i(x_1)$ analytic functions on $\bar I$, $i = 1,2\}$ and

$$\Gamma_i = \{x_1,x_2 | -1 < x_1 < 1,\ x_2 = h_i(x_1)\}$$

$$\Omega_\eta = \Omega \cap S_\eta,\ S_\eta = \{r,\theta | 0 < \theta \le 2\pi,\ 0 < r < \eta,\ \eta > 0\},$$

$$\Omega^*_\eta = S_\eta - \Omega_\eta,$$

where (r,θ) are polar coordinates with the origin at $(-1,0)$. Let $g_1 \in \mathcal{B}^1_{\hat\beta}(\Gamma_1)$, $0 < \hat\beta < 1/2$ (respectively $g_2 \in \mathcal{B}^2_{\hat\beta}(\Gamma_1)$,

$1/2 < \hat{\beta}_i < 1)$, $\beta_1 = \beta_2$, $g_i(-1) = 0$ and let $\Phi = r$. Then there exists $\eta > 0$ and $V_1 \in \mathfrak{B}^2_{\bar{\beta}}(\Omega_\eta)$, $V_1^* \in \mathfrak{B}^2_{\bar{\beta}}(\Omega_\eta^*)$, $\bar{\beta} = \hat{\beta}+1/2+\varepsilon$, $\varepsilon > 0$ arbitrary (respectively $V_2 \in \mathfrak{B}^2_{\bar{\beta}}(\Omega_\eta)$, $V_2 \in \mathfrak{B}^2_{\bar{\beta}}(\Omega_\eta^*)$, $\bar{\beta} = \hat{\beta}-1/2+\varepsilon$) such that $V_i|_{\Gamma_1 \cap \bar{\Omega}_\eta} = g_i$ and $V_i|_{\Gamma_2 \cap \bar{\Omega}_\eta} = 0$.

Proof. Because $h_1(x_1)$ is analytic on $\bar{I}$ it can be analytically extended onto $\tilde{I}_\delta = \{-1-\delta < x_1 < 1+\delta\}$. Then the mapping M : $(x_1,x_2) \longrightarrow (y_1,y_2)$, $y_1 = x_1$, $y_2 = x_2-h_1(x_1)$ is analytic on Ω_η, $\eta = \delta/2$ and $M(\Omega_\eta) = \tilde{\Omega}_\eta$. For η_1 sufficiently small we have $\partial\tilde{\Omega}_\eta \cap S_{\eta_1} = \Gamma_1^* \cup \Gamma_2^*$ where $\Gamma_1^* = \{y_1,y_2 | -1 < y_1 < -1+\eta,\ y_2 = 0\}$, $\Gamma_2^* = \{y_1,y_2 | -1 < y_1 < -1+\bar{\eta}_1,\ y_2 = h_2^*(y_1) = h_2(y_1)-h_1(y_1)$ and $h_2(y_1) > \alpha_1(y_1+1)$. In addition it is easy to see that $|J|,|J^{-1}| < \mu < \infty$ where J is the Jacobian of the mapping M. Because $h_2^*(y_1)$ is analytic on $-1 \le y_1 \le -1+\bar{\eta}_1$ we define $\psi(y_1,y_2) = -y_2 + h_2^*(y_1)$ and $\psi(y_1,y_2)$ has the properties in Lemma 4.18. Using now Corollary 4.4, 4.5, $g_1 \in \mathfrak{C}^1_{\hat{\beta}}(\Gamma_1)$, $g_2 \in \mathfrak{C}^2_{\hat{\beta}}(\Gamma_1)$ and hence using Lemma 4.6, $g_1(M^{-1}(y))|_{y_2=0} \in \mathfrak{C}^1_{\hat{\beta}}(\Gamma_1^*)$, $g_2\ (M^{-1}(y))|_{y_2=0} \in \mathfrak{C}^2_{\hat{\beta}}(\Gamma_1^*)$. Using Lemma 4.8 and Lemma 4.18 there are functions V_1 and V_1^* (respectively V_2 and V_2^*) on $\tilde{\Omega}_\eta \cap S_{\eta_2}$ (respectively $\tilde{\Omega}_\eta^* \cap S_{\eta_2}$), which belongs to $\mathfrak{B}^2_{\hat{\beta}+\varepsilon/2}(\Omega_\eta \cap S_{\eta_2})$ (respectively $\mathfrak{B}^2_{\hat{\beta}+\varepsilon/2}(\tilde{\Omega}_\eta^* \cap S_{\eta_2})$). Using now Lemmas 4.7, 2.3, our lemma follows. $\square$

The lemma leads to the following.

Theorem 4.6. Theorems 4.3 and 4.5 hold also for Lipschitzian curvilinear polygon when $\bar{\beta}_i$ are replaced by $\bar{\beta}_0+\varepsilon$, $\varepsilon > 0$ arbitrary.

Proof. Because the edges are analytic curves and g are analytic on Γ_0 (but not on $\bar{\Gamma}_0$) we show similarly (as in the proof of Theorem 4.1) that the solution u of the Laplace equation belongs to $\mathfrak{B}^2_{\bar{\beta}+\varepsilon}(\Omega)$. This can be done identically as in the proofs of Theorems 3.3 and 3.4 of [6], showing that $u \in \mathfrak{C}^2_{\beta+\varepsilon}(\Omega)$.

<u>Remark 4.7</u>. Comparing the respective theorems for straight and curvilinear polygons we see that in the latter case we are losing slight in the regularity. It is not known whether this loss can be removed.

5. The finite element method

Let us consider the finite element method for solving the model problem (3.1). We will assume that Ω is a curvilinear polygon and for simplicity of the exposition we shall assume that the vertex A_1 is located in the origin and the singularity occurs only in the neighborhood of A_1. In this case we can assume that $\Phi = r$.

Let us first describe the meshes which we will consider. Let $\Omega^n = \{\Omega_{i,j},\ j = 1,\dots,n+1,\ i = 1,\dots,I(j)\}$ be the partition of Ω satisfying the following conditions (see Figure 5.1 where indices i,j of $\Omega_{i,j}$ are given):

(i) $\Omega_{i,j}$ are open quadrilaterals or triangles (curvilinear quadrilaterals or triangles), the intersection of any two $\bar{\Omega}_{i,j}$ is a common vertex or the entire side or is empty (the mesh shown in Figure 5.1 is a geometric mesh with respect to the vertex A_1 (see (iv)). If the singularity would occur also in other vertices then similar refinement would be in the neighborhood of A_j, $j > 1$).

(ii) Let $h_{i,j}$ be the diameter of $\Omega_{i,j}$ and $\underline{h}_{i,j}$ the diameter of the largest circle inscribed in $\Omega_{i,j}$. We shall assume that there is a constant λ independent of n such that

$$h_{i,j}/\underline{h}_{i,j} \le \lambda. \tag{5.1}$$

(iii) Let $M = \{M_{i,j},\ 1 \le i \le I(j),\ i \le j \le n+1\}$ in which $M_{i,j}$ is one to one mapping of the standard (master) square $\bar{S} = [1,1]\times[-1,1]$ respectively standard triangle $T = \{\xi,\eta \mid 0 \le \eta \le 1-\xi,\ -1 \le \xi \le 1\}$ onto $\bar{\Omega}_{i,j}$. If T is a triangle then we will assume that $M_{i,j}$ can be extended into standard square $\bar{S}$ (T is

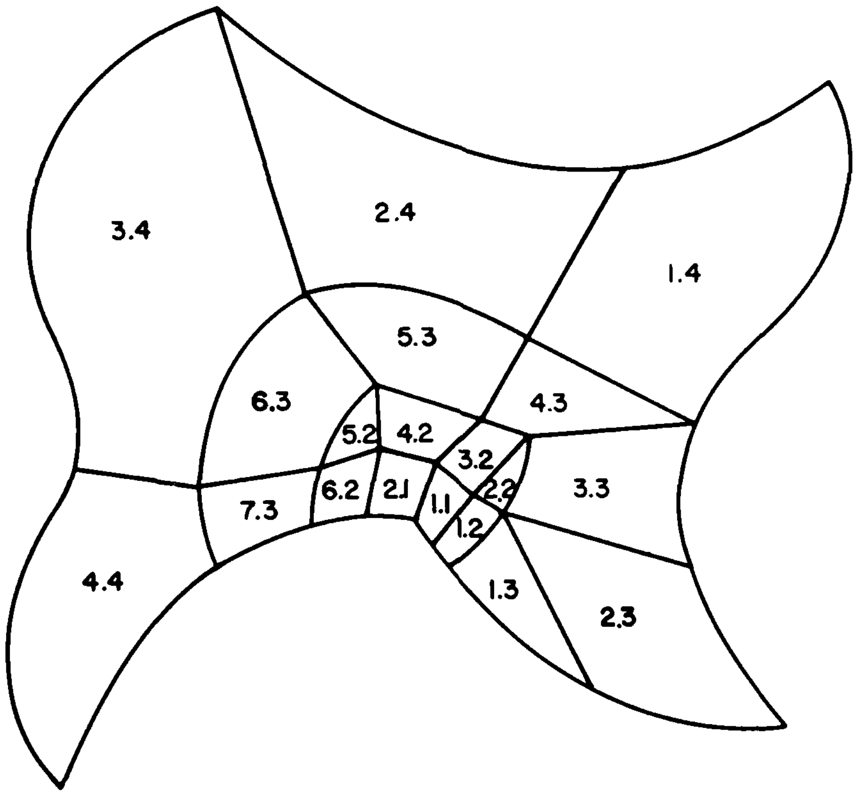

Figure 5.1. Scheme of the mesh.

half of S) such that $M_{i,j}(S) = G_{i,j} \subset \Omega$ and $M_{i,j}$ still satisfies on $G_{i,j}$ all conditions which will be later imposed on $M_{i,j}$. Let $P_{i,j,\ell}$ and $\gamma_{i,j,\ell}$ denote the vertices and sides of $\Omega_{i,j}$; then $M^{-1}_{i,j}(P_{i,j,\ell})$ and $M^{-1}_{i,j}(\gamma_{i,j,\ell})$ are vertices and sides of S, $1 \le \ell \le 4$ (respectively vertices and sides of T with $1 \le \ell \le 3$). Moreover if $M_{i,j}$ and $M_{m,k}$ map (closed) standard square $\bar{S}$ onto element $\bar{\Omega}_{i,j}$ and $\bar{\Omega}_{m,k}$ with the common side $\phi = \overline{P_1,P_2}$ then for any $P \in \gamma$, $\mathrm{dist}(M^{-1}_{i,j}(P), M^{-1}_{i,j}(P_\ell)) = \mathrm{dist}(M^{-1}_{m,k}(P), M^{-1}_{i,j}(P_\ell))$, $\ell = 1,2$.

Let $M_{i,j}(\bar{S}) \cap M_{m,k}(\bar{T}) = \gamma = \overline{P_1,P_2}$ be a common side of the quadrilateral and a triangle. If γ is the image of the sides of the same length we make some assumptions as before. If γ is the image of the sides with different lengths, then we adjust the assumption in the obvious way.

We will assume that the mapping $M_{i,j}$ can be written in the form

$$(5.2)\qquad \begin{aligned} x &= X_{i,j}(\xi,\eta) \\ & \qquad\qquad\qquad (\xi,\eta) \in S \quad (\text{or} \quad T) \\ y &= Y_{i,j}(\xi,\eta) \end{aligned}$$

with $X_{i,j}, Y_{i,j}$ being smooth functions on S (respectively T) and for which more assumptions will be made later. We shall assume that for $|\alpha| \le 2$

$$(5.3)\qquad |D^\alpha X_{i,j}|, |D^\alpha Y_{i,j}| \le C_0 h_{i,j}$$

and

$$(5.4)\qquad C_1 h_{i,j}^2 \le J_{i,j} \le C_2 h_{i,j}^2$$

where C_0, C_1, C_2 are constants independent of i,j and n and $J_{i,j}$ is the Jacobian of the mapping $M_{i,j}$.

The mesh $\Omega_\sigma^n (0 < \sigma < 1)$ is called geometrical mesh with the ratio $\sigma < 1$ with respect to the origin when in addition following conditions are satisfied.

(iv) Let $d_{i,j}$ denote the distance between the origin and quadrilateral $\Omega_{i,j}$; then we assume that

$$(5.5)\qquad c\sigma^{n+2-j} \le d_{i,j} \le \bar{c}\sigma^{n+1-j} \quad \text{for} \quad 1 < j \le n+1, \ 1 \le i \le I(j)$$

$$(5.6)\qquad d_{i,1} = 0 \quad \text{for} \quad 1 \le i \le I(1)$$

$$(5.7)\qquad \varkappa_1 d_{i,j} \le h_{i,j} \le \varkappa_2 d_{i,j}$$

(5.8) $$\varkappa_3 \sigma^{n+1} \le h_{i,1} \le \varkappa_4 \sigma^n$$

where $c, \bar{c}, \varkappa_i$, $i = 1,\dots,4$ are positive constants independent of i,j,n. If $\Omega_{i,j}$ is a <u>triangle</u> then we assume that

(5.9) $$\operatorname{dist}(G_{i,j},0) \ge C \operatorname{dist}(\Omega_{i,j},0)$$

and if γ is the common side of $\Omega_{i,j}$ and $\Omega_{m,\ell}$, $j > \ell$, then γ is the side of $G_{i,j}$; if γ is the common side of $\Omega_{\ell,j}$ and $\Omega_{i,j}$, then γ is the side of $G_{i,j}$ or $G_{\ell,j}$; if γ is the side of $\Omega_{i,j}$ and the part of $\partial\Omega$, then γ is the side of $G_{i,j}$. In Figure 5.2 we show the association of $G_{i,j}$ and $\Omega_{i,j}$. In our example $R(1) = 5$, $R(2) = 12$ as can be seen the numbering is largely arbitrary. $\Omega_{i,j}$ are shadowed by full lines and $G_{i,j}$ by extended dashed lines. The indices i,j are indicated in Figure 5.2.

Let us verify now our assumptions. The condition (5.9) is obviously satisfied. Let $\gamma_1 = \bar{\Omega}_{11,2} \cap \bar{\Omega}_{4,1}$. Then γ_1 is the side of $\bar{G}_{11,2}$ $(= \bar{\Omega}_{11,2} \cup \bar{\Omega}_{10,2})$ which is our condition.

Let $\gamma_2 = \bar{\Omega}_{1,1} \cap \bar{\Omega}_{2,1}$. Then γ_2 is neither side of $G_{1,1}$ nor $G_{2,1}$ and our assumption is not satisfied. In this case we have to define $\bar{G}_{2,1} = \bar{\Omega}_{2,1} \cup \bar{\Omega}_{4,2}$.

In application we can always assume that a proper association between $\Omega_{i,j}$ and $G_{i,j}$ always exists. Nevertheless we remark that our assumptions mentioned above could be difficult to precisely verify, especially that $M_{i,j}$ is one to one mapping. Nevertheless this is common in the finite element practice.

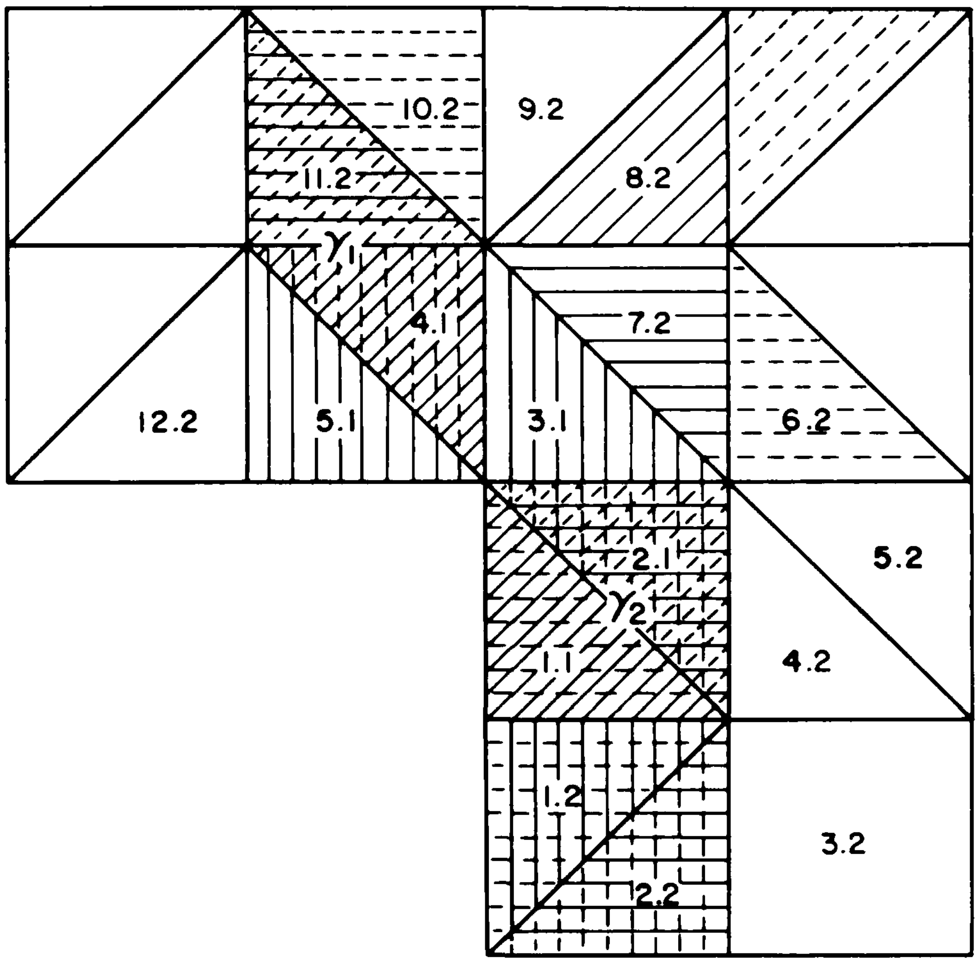

Figure 5.2. The scheme of the mesh, $\Omega_{i,j}$ and $G_{i,j}$.

So far we have assumed that the geometric mesh was refined only in the neighborhood of one vertex (singular point). Analogously we define the geometric mesh in the neighborhood of every or some vertices. Instead of the formal definition we show in Figure 5.4 a geometric mesh for the domain Ω shown in Figure 5.3. The vertices in which neighborhood the mesh should be refined are numbered.

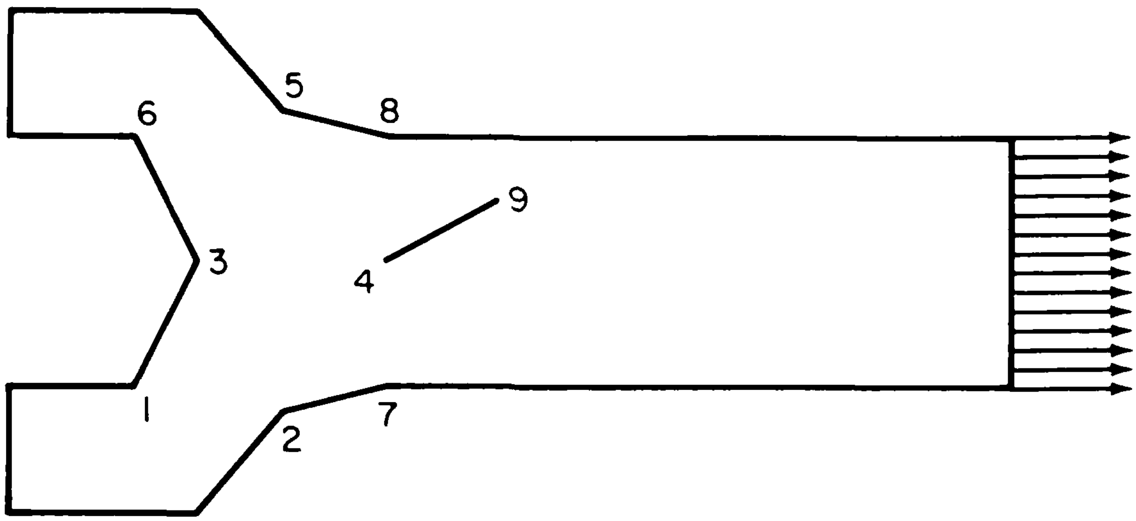

Figure 5.3. The domain Ω.

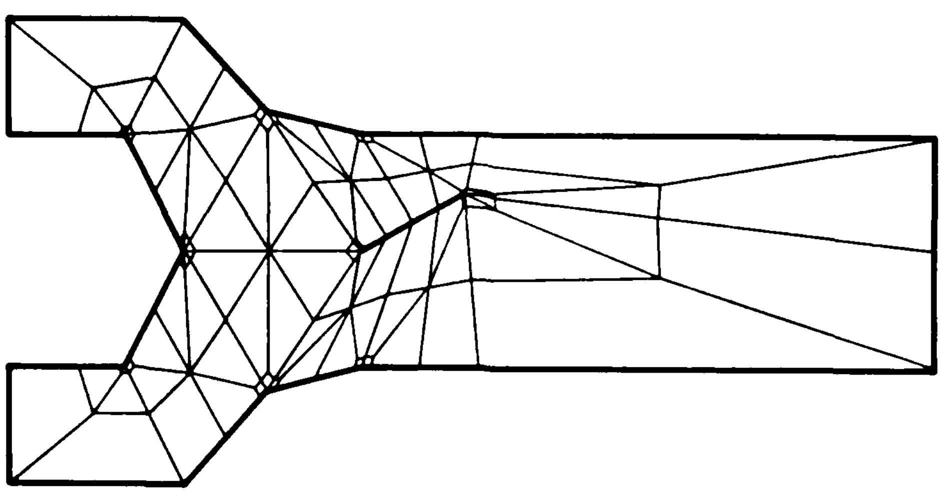

Figure 5.4. Geometric mesh on the domain Ω shown in Figure 5.3.

Let now $\underline{P} = (p_{i,j}, 1 \le i \le R(j), 1 \le j \le n+1)$ and $\underline{Q} = (q_{i,j}, 1 \le i \le R(j), 1 \le j \le n+1)$ be the degree vectors with integers $p_{i,j}, q_{i,j} \ge 0$. We define the subspace $S^{\underline{P},\underline{Q}}(\Omega_\sigma^n) = \{\varphi | \varphi(x_1,x_2) = \Phi(M_{i,j}^{-1}(x_1,x_2))$ for $(x_1,x_2) \in \Omega_{i,j}$, $\Phi_{i,j}(\xi,\eta)$, $\xi,\eta \in S$ (respectively $\xi,\eta \in T$) is a polynomial of degree $\le p_{i,j}$ in ξ and of degree $\le q_{i,j}$ in η, (respectively of total degree $\max(p_{i,j},q_{i,j})\}$. Further we denote $S^{\underline{P},\underline{Q},1}(\Omega_\sigma^n) = S^{\underline{P},\underline{Q}}(\Omega_\sigma^n) \cap H^1(\Omega)$ (usually but not always $p_{i,j} = q_{i,j}$).

Let us impose now additional assumptions on Ω_σ^n. First let us assume that $\Omega_{i,j} \in \Omega_\sigma^n$ are quadrilaterals. In this case let $\gamma_{i,j,\ell}$, $1 \le \ell \le 4$ be the side of the quadrilateral $\Omega_{i,j} \in \Omega_\sigma^n$. Then we assume

$$(5.9a) \qquad \gamma_{i,j,\ell} = \begin{cases} x = h_{i,j}\varphi_{i,j,\ell}(\xi) \\ y = h_{i,j}\psi_{i,j,\ell}(\xi), \end{cases} \qquad -1 \le \xi \le 1, \quad \ell = 1,3$$

$$(5.9b) \qquad \gamma_{i,j,\ell} = \begin{cases} x = h_{i,j}\varphi_{i,j,\ell}(\eta) \\ y = h_{i,j}\psi_{i,j,\ell}(\eta), \end{cases} \qquad -1 \le \eta \le 1, \quad \ell = 2,4$$

and that for some constants $C \ge 1$, $L \ge 1$, which are independent of i,j,ℓ we have

$$(5.10) \qquad |\varphi_{i,j,\ell}^{(k)}|, \ |\psi_{i,j,\ell}^{(k)}| \le CL^k k!, \quad k = 1,2,...$$

and that the mapping $M_{i,j}$ which maps S onto $\Omega_{i,j}$ has the form

$$
(5.10)\quad M_{i,j} = \begin{cases}
x = X_{i,j}(\xi,\eta) = \{\varphi_{i,j,1}(\xi)\,\frac{(1-\eta)}{2} \\
\quad + \varphi_{i,j,2}(\eta)\,\frac{(1+\xi)}{2} + \varphi_{i,j,3}(\xi)\,\frac{(1+\eta)}{2} \\
\quad + \varphi_{i,j,4}(\eta)\,\frac{(1-\xi)}{2}\}h_{i,j} \\
\quad - x_{i,j,1}\,\frac{(1-\xi)}{2}\,\frac{(1-\eta)}{2} - x_{i,j,2}\,\frac{(1+\xi)}{2}\,\frac{(1-\eta)}{2} \\
\quad - x_{i,j,3}\,\frac{(1+\xi)}{2}\,\frac{(1+\eta)}{2} - x_{i,j,4}\,\frac{(1+\eta)}{2}\,\frac{(1-\xi)}{2} \\
y = Y_{i,j}(\xi,\eta) = \{\psi_{i,j,1}(\xi)\,\frac{(1-\eta)}{2} \\
\quad + \psi_{i,j,2}(\eta)\,\frac{(1+\xi)}{2} + \psi_{i,j,3}(\xi)\,\frac{(1+\eta)}{2} \\
\quad + \psi_{i,j,4}(\eta)\,\frac{(1-\xi)}{2}\}h_{i,j} \\
\quad - y_{i,j,1}\,\frac{(1-\xi)}{2}\,\frac{(1-\eta)}{2} - y_{i,j,2}\,\frac{(1+\xi)}{2}\,\frac{(1-\eta)}{2} \\
\quad - y_{i,j,3}\,\frac{(1+\xi)}{2}\,\frac{(1+\eta)}{2} - y_{i,j,4}\,\frac{(1+\eta)}{2}\,\frac{(1-\xi)}{2}
\end{cases}
$$

where we denoted by $(x_{i,j,\ell}, y_{i,j,\ell}) = P_{i,j,\ell}$ the vertices of $\Omega_{i,j}$. The notation is depicted in Figure 5.5 a,b.

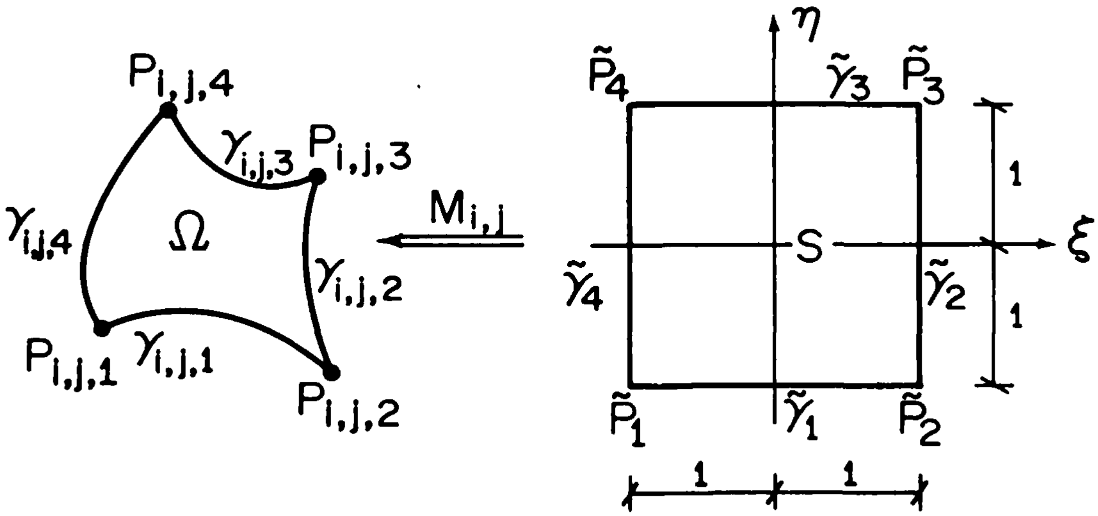

Figure 5.5. The curvilinear quadrilateral $\Omega_{i,j}$ and the standard square S.

In the case that $\Omega_{i,j}$ is a triangle the mapping is essentially similar. We will define it only for the case when only one side is curvilinear. Figure 5.6. shows the notation.

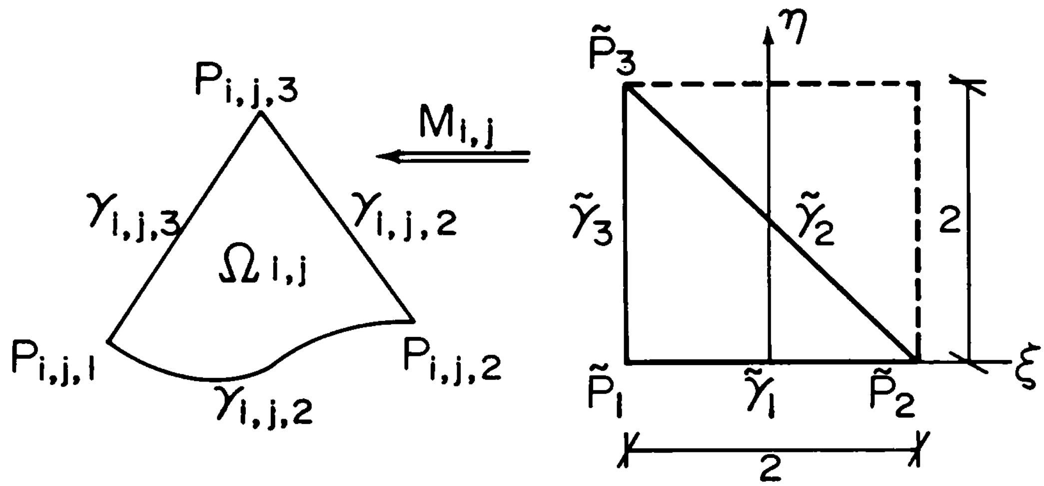

Figure 5.6. The curvilinear triangle $\Omega_{i,j}$ and the standard triangle T.

$$(5.11)\quad M_{i,j} = \begin{cases} x = X_{i,j}(\xi,\eta) = \left\{\left[\varphi(\xi) - x_1 \dfrac{1-\xi}{2} - x_2 \dfrac{1+\xi}{2}\right] \dfrac{1-\xi-\eta}{1-\xi}\right\} h_{i,j} \\ \qquad + x_1\left(\dfrac{1-\xi-\eta}{2}\right) + x_2 \dfrac{\xi+1}{2} + x_3 \dfrac{\eta}{2} \\ \\ y = Y_{i,j}(\xi,\eta) = \left\{\left[\psi(\xi) - y_1 \dfrac{1-\xi}{2} - y_2 \dfrac{1+\xi}{2}\right] \dfrac{1-\xi-\eta}{1-\xi}\right\} h_{i,j} \\ \qquad + y_1\left(\dfrac{1-\xi-\eta}{2}\right) + y_2 \dfrac{\xi+1}{2} + y_3 \dfrac{\eta}{2}. \end{cases}$$

We see that we can extend $M_{i,j}$ onto the standard square S.

Let us now describe the finite element method. It is a standard one.

(a) First given the nonhomogeneous Dirichlet (essential) boundary condition $g^{[0]}$ on $\Gamma^{(0)}$ we project it into the space on traces of the subspace $S^{P,Q,1}(\Omega_\sigma^n) = S$. We denote this projection by g_S, i.e. we replace $g^{[0]}$ on $\Gamma^{(0)}$ by g_S.

(b) The finite element solution $u_S \in S^{\underline{P},Q,1}(\Omega^n_\sigma)$ is now defined in the usual way such that

$$B(u_S,v) = \int_\Omega fvdx + \int_{\Gamma^{(1)}} g^{[1]}vdx$$

holds for all $v \in S^{\underline{P},Q,1}(\Omega^n_S) \cap H^1_0(\Omega)$ when $u_S = g^S$ on $\Gamma^{(0)}$ and

$$(5.12) \quad B(u_S,v) = \int\left[\sum_{i,j=1}^{2} a_{i,j} \frac{\partial u_S}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_{i=1}^{2} b_i \frac{\partial u_S}{\partial x_i} v + cuv\right]dx.$$

We are assuming that $B(u,v)$ satisfies the usual inf sup (B-B) condition (with the positive constant independent of $S^{P,Q,1}(\Omega^n_\sigma)$) on $H^1_0(\Omega)\times H^1_0(\Omega)$.

The projection $g^{[0]} \longrightarrow g_S$ is possible to define in different ways. See [10], [11]. We will use the projection analyzed in [9]. Let $\gamma = M_{i,j}(\tilde{\gamma}) \subset \Gamma^{(0)}$ where $\tilde{\gamma} = (-1 < \xi < 1,\ \eta = 0)$ and let $\tilde{g}(\xi) = g(M_{i,j}(\xi))$. Then $\tilde{g}$ is defined on $\tilde{\gamma}$. (Because we will assume that $g^{[0]} \in \mathcal{B}^1_\beta(\gamma)$, $\beta < 1/2$, it is continuous.) We now define

$$\tilde{g}_S(\xi) = a+b\xi+\tilde{q}_p$$

where $\tilde{q}_p$ is a polynomial of degree $p = p_{i,j}$ in ξ, $\tilde{q}_p(-1) = \tilde{q}_p(1) = 0$ and $\tilde{g}_S(\xi) = \tilde{g}(\xi)$ for $\xi = \pm 1$. Polynomial $\tilde{q}_p$ is now such that

$$(5.13) \qquad \tilde{q}_p' = \sum_{k=1}^{p-1} b_k \ell_k(\xi)$$

where $\ell_k(\xi)$ are Legendre polynomials and b_k are the coefficients of the Legendre expansion of $(\tilde{g}-a-b\xi)'$. We mention that the sum in (5.12) starts with $k = 1$ because

$$\int_{-1}^{1} (\tilde{g}-a-b\xi)'dx = 0.$$

Further we underline that for any $\tilde{g}(\xi)$ which is continuous we define b_k in (5.13) by the integration by parts.

Finally we define $g_S(x) = \tilde{g}_S(M_{i,j}^{-1}(x))$. We have now the following.

<u>Theorem 5.1</u>. Let Ω be a polygon or curved polygon. Assume that the solution u of problem 3.1 belongs to $\mathfrak{C}_{\beta}^{2}(\Omega)$, , $g^{[0]}$ is continuous on $\Gamma^{[0]}$ and $g_i = g^{[0]}/\Gamma_i \in \mathfrak{B}_{\hat{\beta}}^{1}(\Gamma_i)$, $0 < \hat{\beta}_i < 1/2$ or $g_i \in \mathfrak{B}_{\hat{\beta}}^{2}(\Gamma_i)$, $1/2 < \hat{\beta}_i < 1$, $i \in Q$. Let $S = S^{P,Q,1}(\Omega_{\sigma}^{n})$ and u_S is the finite element solution defined above. Let $0 < \mu < \nu < \infty$ and let $\mu j \le p_{i,j} = q_{i,j} < \nu n$, $p_{i,j}, q_{i,j} \ge 1$. Then

$$\|u-u_S\|_{H^1(\Omega)} \le Ce^{-bN^{1/3}} \tag{5.13}$$

where $N = \dim S^{P,Q,1}(\Omega_{\sigma}^{n})$ and the constant C in (5.13) is independent of N.

The proof will be given in the next section.

In Theorem 5.1 we assumed that the solution is in the space $\mathfrak{C}_{\beta}^{2}(\Omega)$. In section 4 we discussed the structure of the input data in the boundary condition functions $g^{[i]}$, $i = 0,1$. $g^{[i]}$ and f guarantee that the solution of problem (3.1) belongs to the space $\mathfrak{C}_{\beta}^{2}(\Omega)$.

<u>Remark 5.1</u>. Assume that $b_i = 0$ and $c > 0$ in (5.12). If

$g^{[0]} = 0$ and $S^{P_2,Q_2,1}(\Omega_\sigma^n) \supset S^{P_1,Q_1,1}(\Omega_\sigma^n)$ then

$$B(u_{S_1}, u_{S_1}) \le B(u_{S_2}, u_{S_2}) \tag{5.14}$$

where $u_{S_i} \in S^{P_i,Q_i,1}(\Omega_\sigma^n)$ is the finite element solution. If $g^{[0]} \neq 0$ then (5.14) does not hold in general. If $g^{[1]} = 0$ and $f = 0$ and $g^{[0]}$ belongs to the space of traces of $S^{P_i,Q_i,1}(\Omega_\sigma^n)$, $i = 1,2$, (i.e, $g_{S_1} = g_{S_2}$) then

$$B(u_{S_1}, u_{S_1}) \ge B(u_{S_2}, u_{S_2}). \tag{5.15}$$

If $g^{[0]}$ does not belong to the space of both $S^{P_i,Q_i,1}(\Omega_\sigma^n)$, $i = 1,2$, then (5.15) does not hold in general (although numerical experience shows that in most cases (5.15) still holds).

6. The rate of convergence

We will prove in this section the statement of Theorem 5.1. Let us prove some auxiliary lemmas.

Lemma 6.1. Let $g \in H^k(I)$, $k \geq 1$, $g(-1) = g(1) = 0$ and let $p \geq 2$. Let

$$g'(x) = \sum_{j=0}^{\infty} a_j \ell_j(x) \tag{6.1}$$

where $\ell_j(x)$, $j = 0,1,\ldots$ is the Legendre polynomial. Let

$$g_p'(x) = \sum_{j=1}^{p-1} a_j \ell_j(x) \tag{6.2}$$

$$g_p(x) = \int_{-1}^{x} g_p'(x)dx. \tag{6.3}$$

Then

$$g_p(\pm 1) = 0 \tag{6.4}$$

and

$$\|g^{(m)} - g_p^{(m)}\|^2_{L_2(I)} \leq C \frac{1}{p^{2(1-m)}} \frac{\Gamma(p-s+1)}{\Gamma(p+s+1)} \|g^{(s+1)}\|^2_{L_2(I)} \tag{6.5}$$

$$p \geq s \geq 0, \; m = 0,1.$$

Proof. Because $g' \in L_2(I)$ expansion (6.2) exists and because $g(\pm 1) = 0$, $a_0 = 0$ and (6.4) holds. Further obviously

$$\|g' - g_p'\|_{L_2} = \sum_{j=p}^{\infty} |a_j|^2 \frac{2}{2j+1}.$$

We have

$$\frac{d^s}{dx^s} \ell_j(x) = \frac{1}{2^s} \frac{\Gamma(j-s+1)}{\Gamma(j+1)} P^{s,s}_{j-s}(x), \quad s \leq j$$

where $P_j^{s,s}(x)$ is a Jacobi polynomial with

$$\int_{-1}^{1}(1-x^2)^s P_j^{s,s}(x)P_k^{s,s}(x)dx = \begin{cases} 0 \quad \text{for} \quad j \neq k \\ \dfrac{s^{2s+1}\Gamma^2(s+j+1)}{(2s+2j+1)\Gamma(j+1)\Gamma(2s+j+1)} \\ \qquad\qquad \text{for} \quad j = k. \end{cases}$$

Hence we get

$$\begin{aligned} \|g^{(s+1)}\|^2_{L_2(I)} &= \|\sum_{j=s}^{\infty} a_j \ell_j^{(s)}(x)\|^2_{L_2(I)} \\ &\geq \int_{-1}^{1}(1-x^2)^s\left[\sum_{j=s}^{\infty} a_j \ell_j^{(s)}(x)\right]^2 dx \\ &\geq \sum_{j=p}^{\infty}|a_j|^2 \frac{2}{2j+1}\frac{(j+s)!}{(j-s)!} \end{aligned}$$

which yields

$$\begin{aligned} \|g'-g_p'\|^2_{L_2(I)} &\leq \frac{(p-s)!}{(p+s)!}\sum_{j=p}^{\infty}|a_j|^2 \frac{2}{2j+1}\frac{(j+s)!}{(j-s)!} \\ &\leq \frac{(p-s)!}{(p+s)!}\|g^{s+1}\|^2_{L_2(I)} \end{aligned}$$

and we get (6.5) for $m = 1$. Further we have

$$g-g_p = \sum_{j=p}^{\infty} a_j(\ell_{j+1}-\ell_{j-1})\frac{1}{2j+1}$$

which immediately leads to (6.5) with $m = 0$.

<u>Lemma 6.2</u>. Let $g \in H_\beta^{1,1}(I)$, $0 < \beta < 1/2$ (respectively $g \in H_\beta^{2,2}(I)$, $1/2 < \beta < 1$) an g_p be defined by (6.2). Then

$$\|g_p'\|_{L_2(I)} < Cp\|g\|_{H_\beta^{1,1}(I)}$$

$$\|g_p\|_{L_2(I)} \le C\ell gp\|g\|_{H^{1,1}_\beta(I)}$$

respectively

$$\|g_p'\|_{L_2(I)} \le Cp\|g\|_{H^{2,2}_\beta(I)}$$

$$\|g_p\|_{L_2(I)} \le C\ell gp\|g\|_{H^{2,2}_\beta(I)}.$$

<u>Proof</u>. We have for $0 < \beta < 1/2$ and $g \in H^{1,1}_\beta(I)$

$$|a_j| \le \frac{2j+1}{2} \left|\int_{-1}^{1} g'(x)\ell_j(x)dx\right|$$

$$\le \frac{2j+1}{2} \|g\|_{H^{1,1}_\beta(I)} \left[\int_{-1}^{1} |\ell_j(x)|^2 \hat{\Phi}_\beta^{-2}(x)dx\right]^{1/2}$$

$$\le C\frac{2j+1}{2}\|g\|_{H^{1,1}_\beta(I)}$$

because $|\ell_j(x)| \le 1$. Let now $g \in H^{2,2}_\beta(I)$, $1/2 < \beta < 1$. Then also

$$|a_j| \le \frac{2j+1}{2}\left|\int_{-1}^{1} g'(x)\ell_j(x)dx\right|$$

$$\le C\frac{2j+1}{2} \|g\|_{H^1(I)} \le C\frac{2j+1}{2}\|g\|_{H^{2,2}_\beta(I)}.$$

Hence

$$\|g_p'\|^2_{L_2(I)} \le \sum_{k=1}^{p-1} |a_k|^2 \frac{2}{2k+1} \le Cp^2\|g\|^2_{H^{1,1}_\beta(I)}$$

(respectively $(Cp^2\|g\|^2_{H^{2,2}_\beta(I)})$, and

$$\|g_p\|^2_{L_2(I)} \le C\sum_{k=1}^{p} |a_k|^2 \frac{2}{(2k+1)^3} \le C|\ell gp|^2\|g\|^2_{H^{2,2}_\beta(I)}.$$

Proof of Theorem 5.1. The basic idea of the proof is very similar to the proof of Theorem 5.3 of [6]. Hence we will outline only the basic steps and underline the essential differences in detail. For simplicity and without any loss of generality we shall assume that there is a singularity in one vertex A_1 only which is placed in the origin; we did make the same assumption also in [6].

We shall first assume that the mesh consists only of the quadrilaterals and that $p_{i,j} = q_{i,j} = p_j \geq 1$. The proof has few steps similar to those in [6],[13].

Step 1. Denote $U_{i,j}(\xi,\eta) = u(M_{i,j}(\xi,\eta))$. Then

(i) for $j > 1$, $U_{i,j}$ is analytic on $\bar{S}$.

(ii) for some constants d and c independent of i,j, $i = 1,\ldots,R(j)$, $j = 1,2,\ldots,n+1$, and $|\alpha| = k$, $k = 1,2,\ldots$ we have

$$(6.6) \qquad |D^{\alpha}U_{i,j}| \leq Ck!d^{k}\sigma^{1-\beta(n-j+2)}.$$

The proof is given in Lemma 5.1 of [6].

Let γ_{ℓ}, $\ell = 1,\ldots,4$ be the sides of S. Assume that γ_1 lies on ξ axis (i.e., $\gamma_1 = I$) and $v(\xi)$ is a polynomial of degree p on γ_1 and vanishes at the end points of γ_1. Then there exists polynomial $V(\xi,\eta)$ of degree p in ξ and η such that

$$(6.7a) \qquad V(\xi,\eta)|_{\gamma_1} = v(\xi),\ V(\xi,\eta)|_{\gamma_{\ell}} = 0,\ \ell = 2,3,4$$

and

$$(6.7b) \qquad \|V\|_{H^1(S)} \leq C|v|_{H^2(\gamma_1)}.$$

For proof see Lemma 5.2 of [6] or [13].

Step 2. We construct polynomial $\tilde{\Phi}_{i,j}(\xi,\eta)$ of degree p_j in ξ and η on S such that $\ell_{i,j} = \tilde{\Phi}_{i,j}$ at the vertices of S and for $m = 0,1,2$, $1 \le s_j \le p_j$.

$$(6.8a) \qquad \|D^m(U_{\ell,j}-\tilde{\Phi}_{i,j})\|^2_{H^0(S)} \le C \frac{(p_j-s_j)!}{(p_j+s_j+2-2m)!} \|U_{i,j}\|^2_{H^{s_j+3}(S)}$$

and using (6.6)

$$\le C \frac{(p_j-s_j)!}{(p_j+s_j+2-2m)!} [d^{s_j+3}(s_j+3)!]^2 \sigma^{2(1-\beta)(n-j+2)}$$

(for the proof see Lemma 4.2 of [13]). Define $\varphi_{i,j}(x,y) = \tilde{\Phi}_{i,j}(M^{-1}_{i,j}(x_1,x_2))$ for $j \ge 2$; then we have for $0 \le m \le 1$

$$\|D^m(u-\varphi_{i,j})\|_{H^0(\Omega_{i,j})} \le Ch^{1-m}_{i,j}\|D^m(U_{\ell,j}-\tilde{\Phi}_{i,j})\|_{H^0(s)}$$

$$\le Ch^{1-m}_{i,j}\left[\frac{(p_j-s_j)!}{(p_j+s_j+2-2m)!}\right]^{1/2} d^{s_j+3}(s_j+3)!\sigma^{(1-\beta)(n+j+2)}.$$

For $j = 1$ we use $p = 1$ and get

$$(6.9a) \qquad \|U_{i,1}-\tilde{\Phi}_{i,1}\|_{H^1(S)} \le Ch^{1-\beta}_{i,1}\|u\|_{H^{2,2}_\beta(\Omega)}$$

$$(6.9b) \qquad \|U_{i,1}-\tilde{\Phi}_{i,1})\|_{H^{2,2}_\beta(S)} \le C\|u\|_{H^{2,2}_\beta(\Omega)}$$

$$(6.9c) \qquad \|u-\varphi_{i,1}\|_{H^1(\Omega_{i,1})} \le Ch^{1-\beta}_{i,1}\|u\|_{H^{2,2}_\beta(\Omega)}$$

in (6.9b) we define $H^{2,2}_\beta(S)$ with the weight $\Phi = r$ with respect to $(-1,-1)$ and we assume that $M_{i,1}((-1,1)) = (0,1) = A_1$.

Step 3. The function $\varphi_{i,j}$ are constructed separately on every $\Omega_{i,j}$ (hence the function φ composed from $\varphi_{i,j} \notin H^1(\Omega)$). Let us assume now that $\gamma = \bar{\Omega}_{i,j} \cap \bar{\Omega}_{k,\ell}$. Then $(\varphi_{i,j}-\varphi_{k,\ell})|_\gamma = \psi \neq 0$; nevertheless $\psi = 0$ in the end points of γ. Let us assume first that $j \geq \ell \geq 2$ and that $\gamma = M_{i,j}(I)$. Denote $\psi(\xi) = \psi(M_{i,j}(\xi))$. Then $\psi(\xi)$ is a polynomial of degree $\bar{p} = \max(p_j,p_\ell)$ on I vanishing at ± 1. Using the imbedding theorem we get

$$\|\psi\|_{H^1(I)} \leq C \max(\|U_{i,j}-\tilde{\Phi}_{i,j}\|_{H^2(S)}, \|U_{k,\ell}-\tilde{\Phi}_{k,\ell}\|_{H^2(S)})$$

and hence by (6.7) there is a polynomial $V(\xi,\eta)$ of degree $\bar{p}$ such that $\tilde{V}_{i,j}(\xi,\eta) = \tilde{\psi}$ on I and

$$\|\tilde{V}_{i,j}(\xi,\eta)\|_{H^1(S)} \leq C\|\tilde{\psi}\|_{H^1(I)}.$$

We estimate then $\|\tilde{\psi}\|_{H^1(I)}$ by (6.8) for $m = 2$. For $j = \ell = 1$ function $\psi = 0$ on γ. For $j = 2$ and $\ell = 1$ we proceed similarly using (6.9b). In this way we construct correction function $V_{i,j}(\xi,\eta)$ and $V_{i,j}(x_1,x_2) = \tilde{V}_{i,j}(M^{-1}_{i,j}(x_1,x_2))$ so that $\varphi_{i,j}+V_{i,j}$ are continuous on every $\gamma \subset \Gamma$, $\gamma = \bar{\Omega}_{i,j} \cap \bar{\Omega}_{k,\ell}$, i.e., the composed function φ such that $\varphi|_{\Omega_{i,j}} = \varphi_{i,j}+V_{i,j}$ belong to $H^1(\Omega)$ and

$$\|u-\varphi\|^2_{H^1(\Omega)} \leq C\Big[\|u\|^2_{H^{2,2}_\beta(\Omega)} \sum_{i=1}^{R(1)} h_{i,1}^{2(1-\beta)}$$

$$+ \sum_{j=2}^{n+1} \sum_{i=1}^{R(1)} \frac{(p_j-s_j)!}{(p_j+s_j-2)!} (d^{s_j+3}(s_j+3)!)^2 \sigma^{2(1-\beta)(n+2-j)}\Big].$$

Step 4. We estimate now $u-\varphi$ at the boundary $\Gamma^{(0)}$. Let $\gamma = \bar{\Omega}_{i,j} \cap \Gamma^{(0)}$. Assume first that $j > 2$. Then using (6.5) and the assumption about $g^{[0]}$ we can construct in the similar way as before the correction function $V_{i,j}$ so that (affix the correction) $\varphi = g_S$ on $\Gamma^{[0]} - \Gamma^{[0]} \cap \bigcup_{j=1} \bar{\Omega}_{i,j}$ and (6.10) still holds.

If $\gamma = \bar{\Omega}_{i,1} \cap \Gamma^{[0]}$ then we use Lemma 6.2 and the assumption that $p \le \nu n$ and analogously as before we construct the correction function $V_{i,1}$ so that function $\hat{\varphi} \in H^1(\Omega)$ is constructed which has the following properties:

(i) $\hat{\varphi} \in H^{P,Q,1}(\Omega_\sigma^n)$.

(ii) $\hat{\varphi} = g_S$ on $\Gamma^{(0)}$

(iii)

$$\|u-\hat{\varphi}\|^2_{H^1(\Omega)} \le C\Big[n^2\sigma^{2(1-\beta)n}$$

$$+ \sum_{j=2}^{n+1} \frac{(p_j-s_j)!}{(p_j+s_j-2)!} (d^{s_j+3}(s_j+3)!)^2\sigma^{2(1-\beta)(n+2-j)}\Big]. \tag{6.11}$$

In (6.11) we have used the assumption about the mesh, namely that $R(j) < K$ independently of n.

<u>Step 5</u>. So far we have not chosen in (6.11) the values of s_j. By the same procedure as in [6] we can select s_j in dependence on p_j so that

$$\|u-\hat{\varphi}\|_{H^1(\Omega)} \le C_1 e^{-bn} \tag{6.12}$$

and because $N < K(\nu n)^2 n < K_1 n^3$ we get

$$\|u-\hat{\varphi}\|_{H^1(\Omega)} \le C_1 e^{-\gamma N^{1/3}} . \tag{6.13}$$

<u>Step 6</u>. Let now $\breve{u}$ be the <u>exact</u> solution of problem (3.1) such that $g^{[0]}$ is replaced by g_S. Then $\breve{u}_S = \breve{u}-u$ satisfies

$$Lu_S = 0$$

$$\frac{\partial u_S}{\partial n_C} = g^{[1]} \quad \text{on} \quad \Gamma^{(1)}$$

$$\tilde{u}_S = g_S - g^{[0]} \quad \text{on} \quad \Gamma^{(0)}.$$

As in Step 4 we construct function $v \in H^1(\Omega)$ such that

$$v = g_S - g^{[0]} \quad \text{on} \quad \Gamma^{(0)}$$

and

$$\|v\|_{H^1(\Omega)} \le C\|u-\hat{\varphi}\|_{H^1(\Omega)}.$$

Because we have assumed that the bilinear form associated to problem (3.1) satisfies the inf-sup (B-B) condition we get immediately

$$\|\tilde{u}_S\|_{H^1(\Omega)} \le C\|u-\hat{\varphi}\|_{H^1(\Omega)}. \tag{6.14}$$

<u>Step 7</u>. Finite element solution u_S can now be understood as the finite element solution of the problem with exact solution $\tilde{u} = u + \tilde{u}_S$. Now we have using (6.13) and (6.14)

$$\|\tilde{u}_S\|_{H^1(\Omega)} \le Ce^{-\gamma N^{1/3}}$$

and hence also

$$\|\tilde{u}-u_S\|_{H^1(\Omega)} \le Ce^{-\gamma N^{1/3}}.$$

Therefore

$$\|u-u_S\|_{H^1(\Omega)} \le Ce^{-\gamma N^{1/3}}$$

which was to prove.

So far we have assumed that the mesh consists only of the quadrilaterals. If the mesh has also triangular elements we proceed very analogously.

In Step 1 we use the mapping $M_{i,j}$ which is extended on S and consider $U_{i,j}$ as the image of u on $G_{i,j}$. it is easy to show that the extension function $V(\xi,\eta)$ having the same properties as mentioned in (6.7) exists for T. See e.g. [6].

All other steps are now the same only rendering that the "correction" fucntions now could be of degree $2p_j$ because $\Phi_{i,j}$ is polynomial of degree $2p$ on the diagonal of S.

7. Numerical Examples

Let us consider the problem

$$\Delta u = 0 \quad \text{on} \quad \Omega \tag{7.1}$$

when Ω is an L-shaped domain as shown in Figure 7.1 and the Dirichlet conditions are prescribed on one part of $\partial\Omega$ and the Neumann conditions are the other part of $\partial\Omega$.

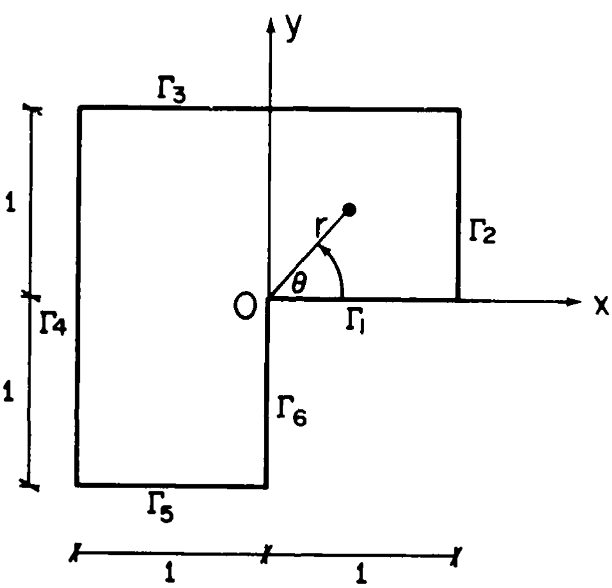

Figure 7.1. The domain Ω.

We will consider two problems with various combinations of the Dirichlet and Neumann boundary conditions with the exact solution.

Case A:

$$u = r^{1/3}\sin \tfrac{1}{3}\theta \tag{7.1}$$

Case B:

$$u = r^{2/3}\cos \tfrac{2}{3}\theta \tag{7.2}$$

The sequence of meshes Ω_σ^n $(\sigma = 0.15)$ is characterized by the parameter $n = 1,\dots,6$ and is shown in Figure 7.2.

Let us first consider the case where the Dirichlet boundary condition is prescribed on the <u>entire</u> $\partial\Omega$. Then in the case A we have $g_i = g|_{\Gamma_i}$ analytic on $\bar{\Gamma}_i$, $i = 1,2,\dots,5$ while on $\Gamma_6 : g_6 = r^{1/3}$. Now we can use Lemma 4.17 and conclude that for $i = 1,\dots,5$, $g_i \in \mathcal{B}^2_\beta(\Gamma_i)$, $0 < \beta < 1$ arbitrary and for $i = 6$ we

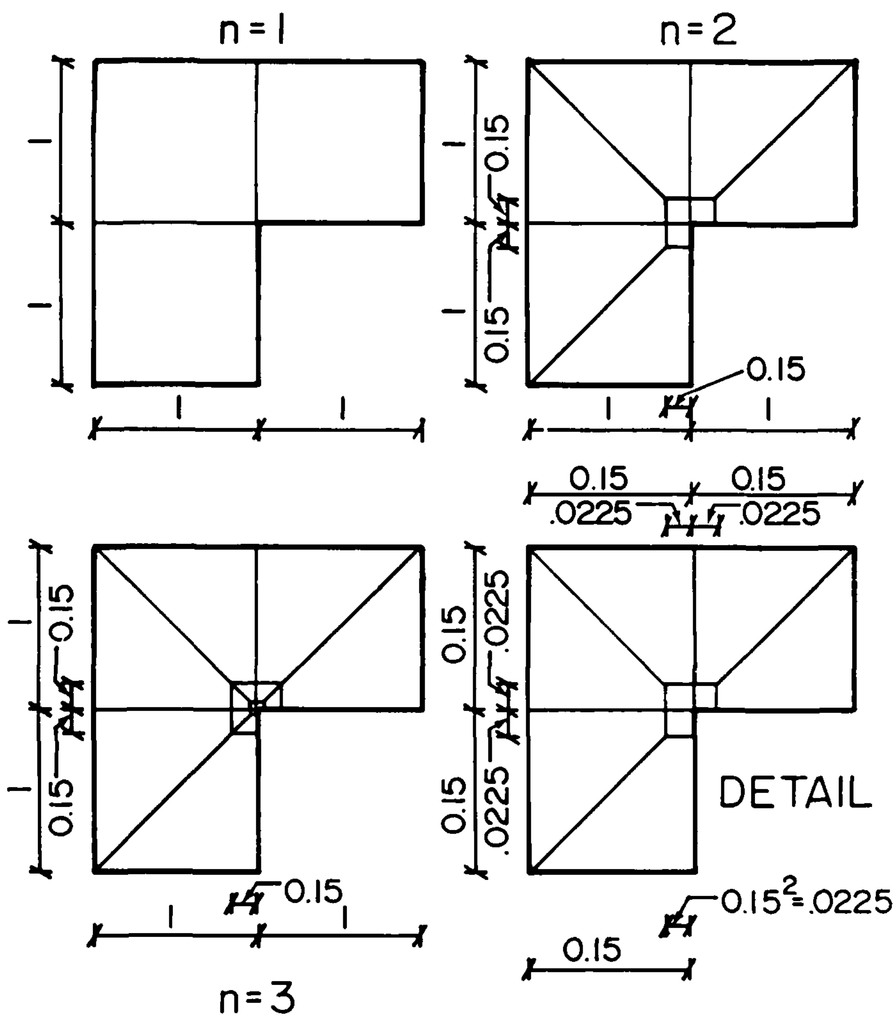

Figure 7.2. The meshes Ω^n_σ, $n = 1,2,3$, $\sigma = 1.5$

obviously have $\nu = \frac{1}{3}$ and hence $g_1 \in \mathcal{B}^1_\beta(\Gamma_6)$ with $\beta > \frac{1}{6}$. Using Theorem 4.3 we see that $g \in \mathcal{B}^{3/2}_\beta(\partial\Omega)$ with $\beta > \frac{2}{3}$ and hence by Theorem 3.2 we have $u \in \mathcal{B}^2_\beta(\Omega)$ with $\beta_1 > \max(\frac{2}{3},\bar{\beta}_1)$ and $\beta_i > \bar{\beta}_i$ for $i = 2,\dots,6$, $\bar{\beta}_i$ depend on the problem. In our case it can be shown that $\bar{\beta}_1 = \frac{1}{3}$ and $\bar{\beta}_i = 0$, $i = 2,\dots,5$. Hence $1 > \beta_1 > \frac{2}{3}$ arbitrary and $0 < \beta_i < 1$ arbitrary for $i = 2,\dots,6$. This of course is obvious also from the fact that the solution is given by (7.1). Analogously in the case B we have $1 > \beta_1 > \frac{1}{3}$, and $0 < \beta_i < 1$ for $1 = 2,\dots,6$.

Denote $E(u) = \frac{1}{2}B(u,u)$, $E_{FE}(n,p) = E(U_S)$ where S being characterized by the mesh Ω^n_σ and degree p and

$$B(u,v) = \int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) dxdy.$$

Let $e = u-u_S$, $\|e\|^2_E = E(e)$ and $\|e\|_{ER} = \|e\|_E/\|u\|_E$ be the error, the error norm and the relative error norm, respectively.

We have in the case A: $E(u) = 0.423569$ and in the case B: $E(u) = 0.918113$.

Figure 7.3a,b depict the relative error of the finite element solution in Ω^n_σ in the double logarithmic scale. We mention that for $n = 1$ and $p = 1$ we have $N = 0$ because the finite element solution is determined directly by its values at the boundary. We see that in the case A the p-version does not practically converge while in the case B it does. Nevertheless the h-p version $(n = p)$ converges in both cases as the theorems 5.1 predicts. We also show in the figure the degree of the elements. Figure 7.4a,b

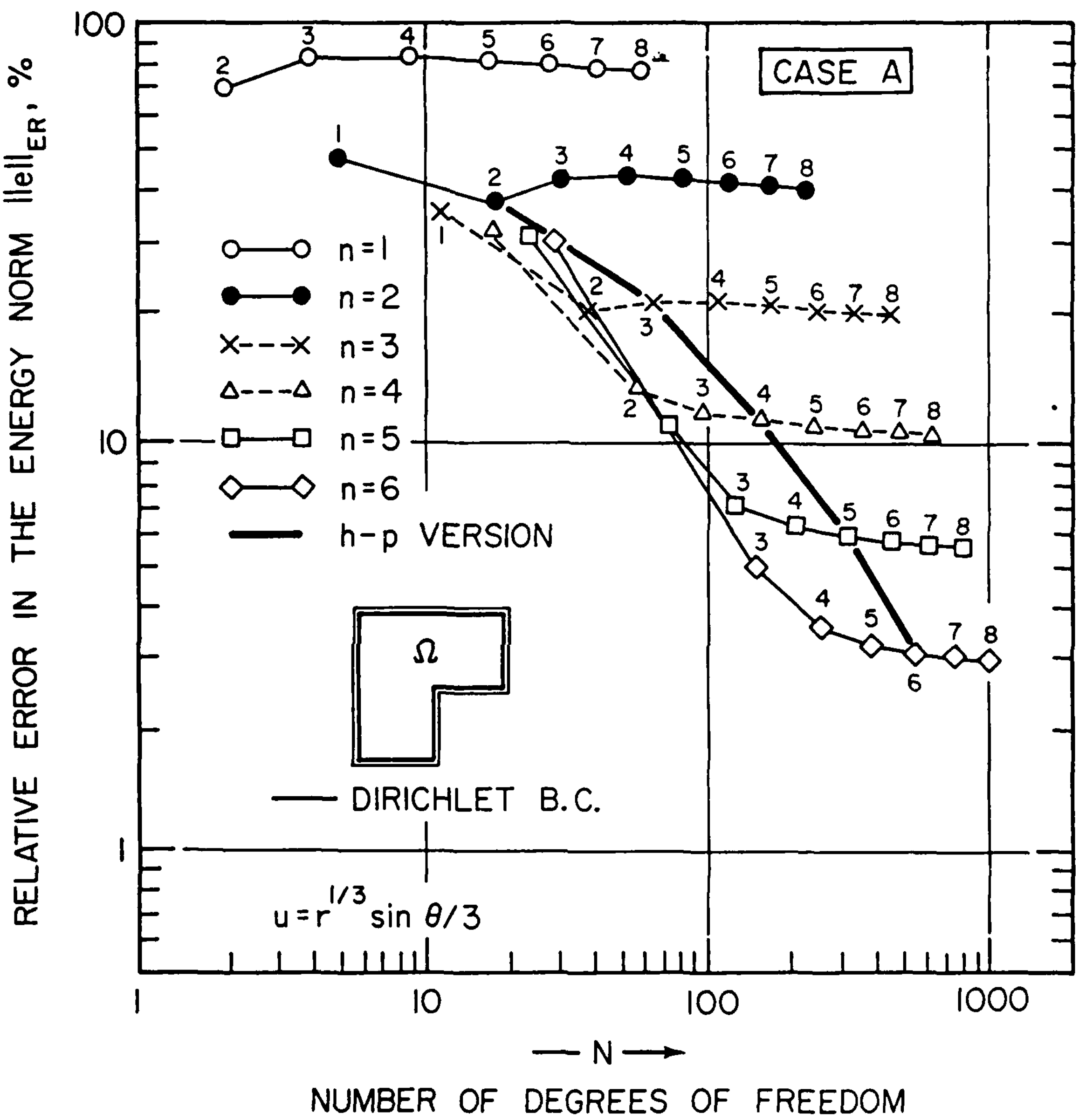

Figure 7.3a. The relative error of the p and h-p version in $\ell g \, \|e\|_{ER} \times \ell g \, N$ scale, Case A.

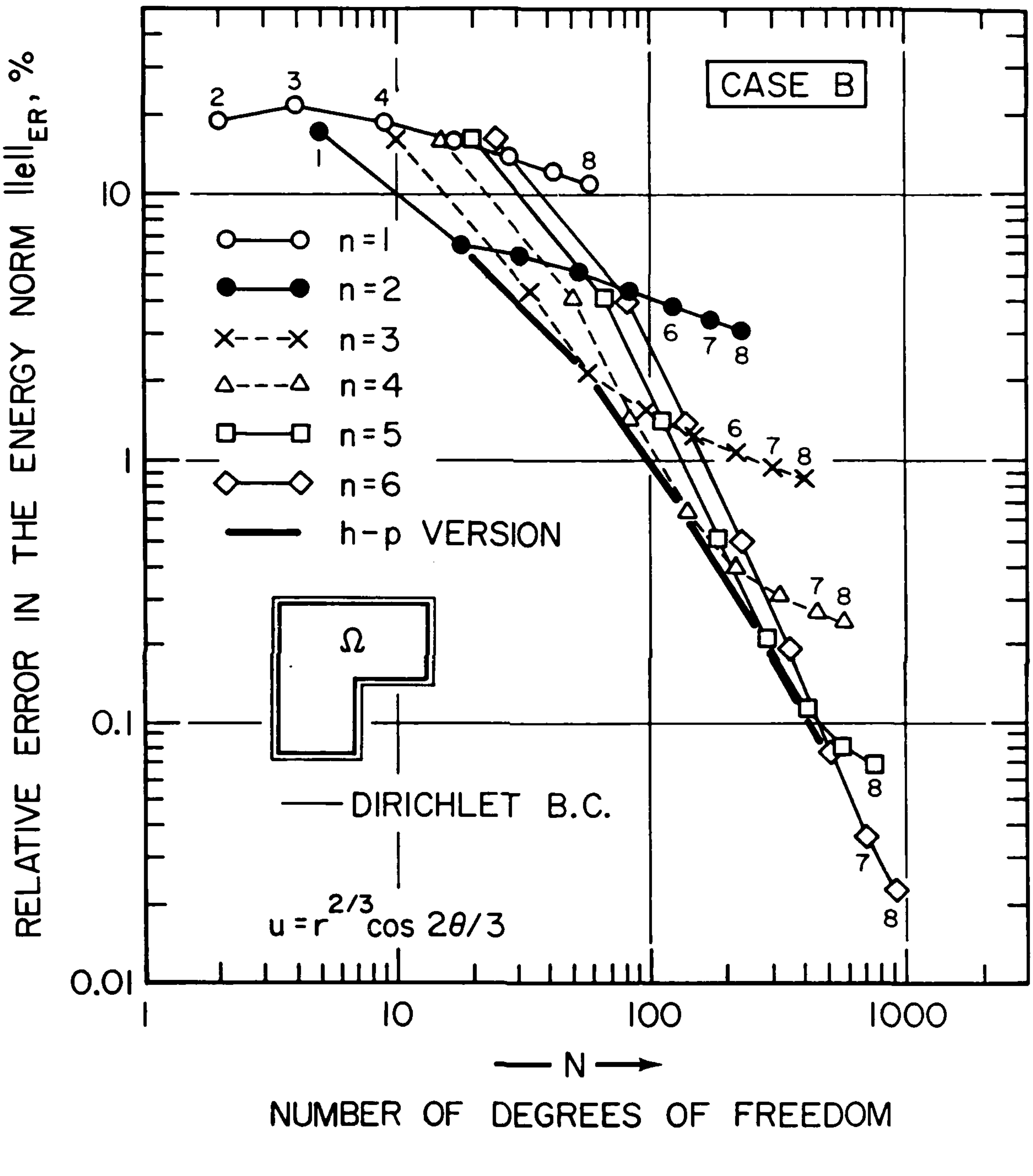

Figure 7.3b. The relative error of the p and h-p version in $\ell g\,||e||_{ER}$ - $\ell g\,N$ scale, Case B.

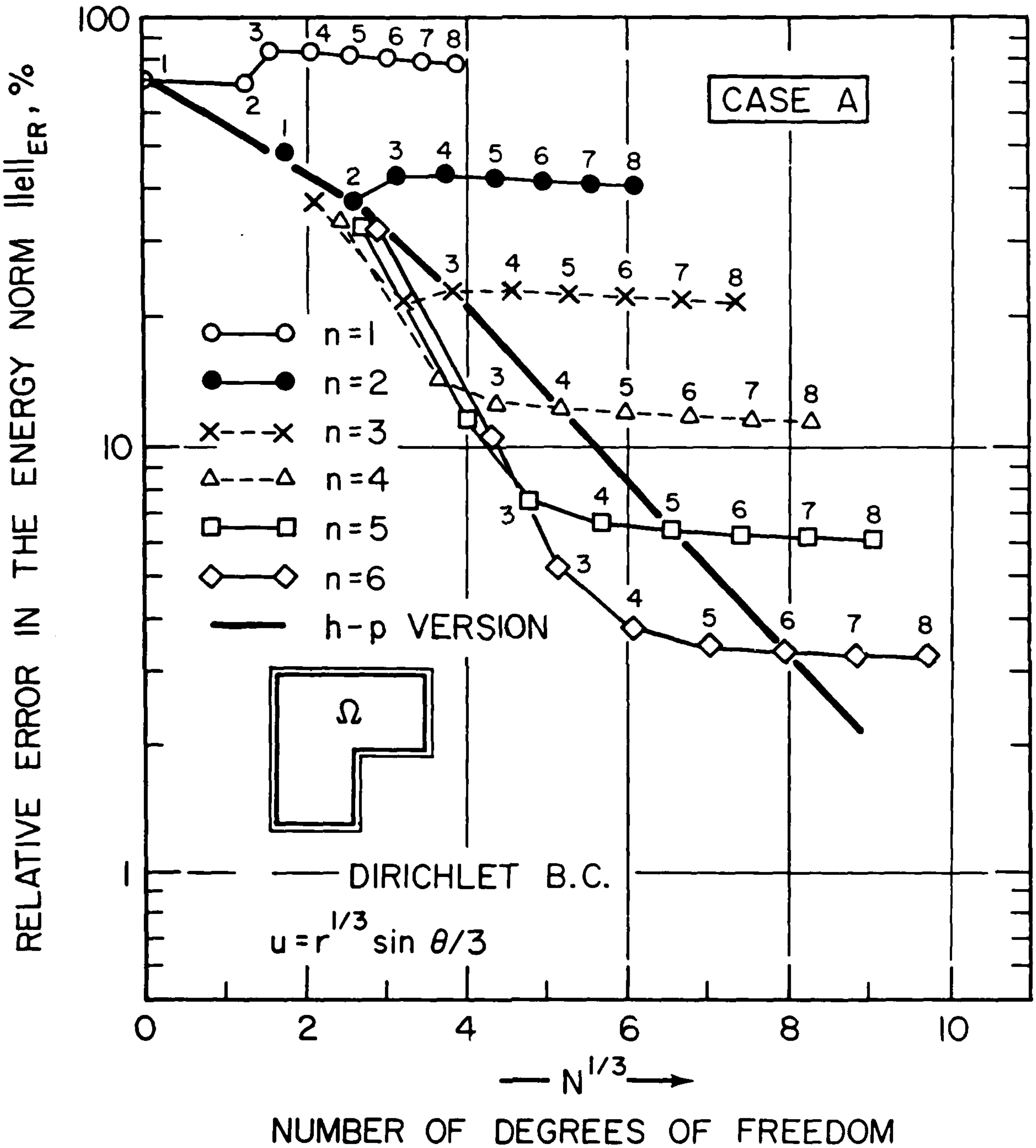

Figure 7.4a. The relative error of the p and h-p version in $\ell g \, \|e\|_{ER} \times N^{1/3}$ scale, Case A.

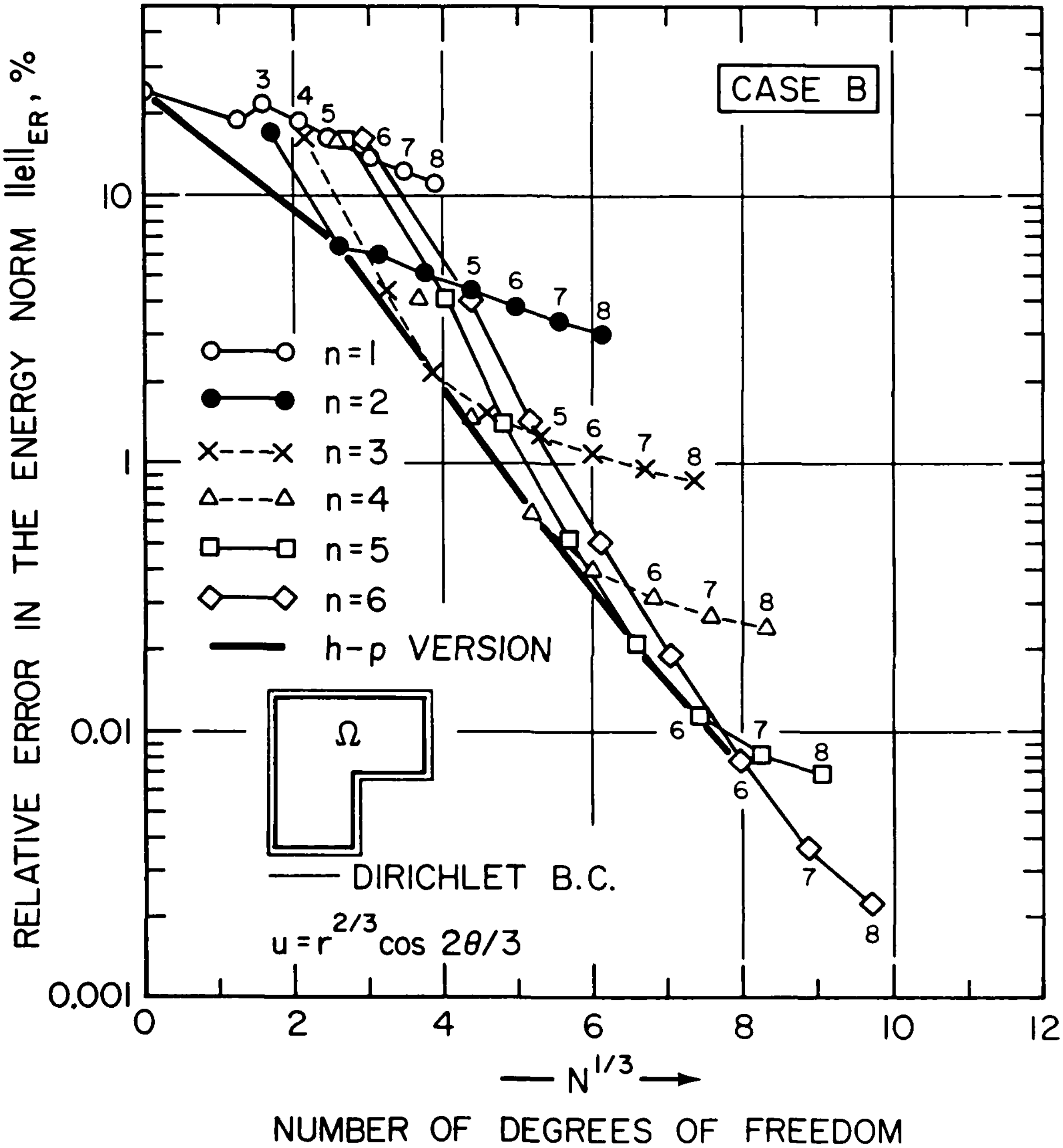

Figure 7.4b. The relative error of the p and h-p version in $\ell g \|e\|_{ER} \times N^{1/3}$ scale, Case B.

show the same results but in the scale $\ell g\|e\|_{ER} \cdot N^{1/3}$. We see that the error of the h-p version is nearly linear in this scale which shows that the error decreases exponentially as $O(e^{-\alpha\sqrt[3]{N}})$.

The divergence of the p-version in the case A is related to the way how g is replaced by g_S (see Lemma 6.2). We have shown in [9] that the p-version converges <u>with optimal rate</u> provided that $g \in H^1(\Gamma)$. If $g \notin H^1(\Gamma)$ to get optimal rate of convergence we have to replace g by g_S in another way (so called $H^{1/2}$ projection, see [10]). Then the convergence is also guaranteed. Figure 7.5 compares the performance of the method with $H^{1/2}$ projection in the case A for $n = 1$. We show in Figure 7.5 the slope of $O(N^{-1/3})$ which is also the optimal rate. For the detailed comparison of the performance of various projections, specifically the H^1 and $H^{1/2}$-projection we refer to [7].

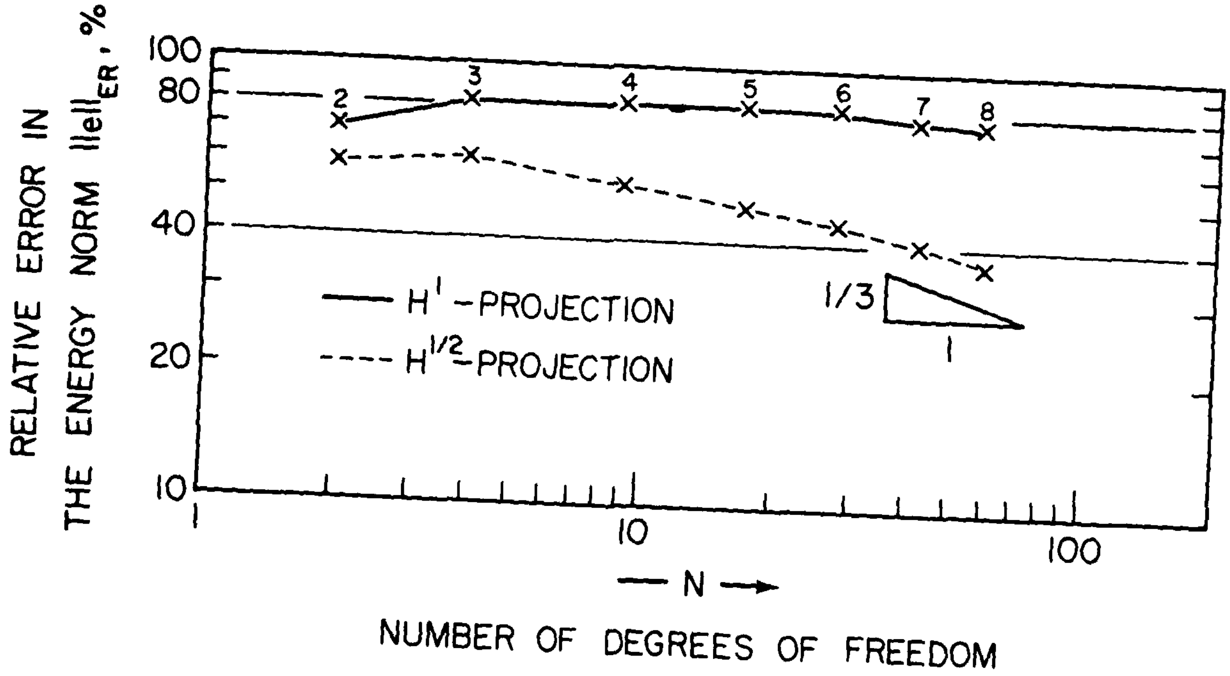

Figure 7.5. Relative error of the p-version for the H^1-projection and $H^{1/2}$-projection, Case A.

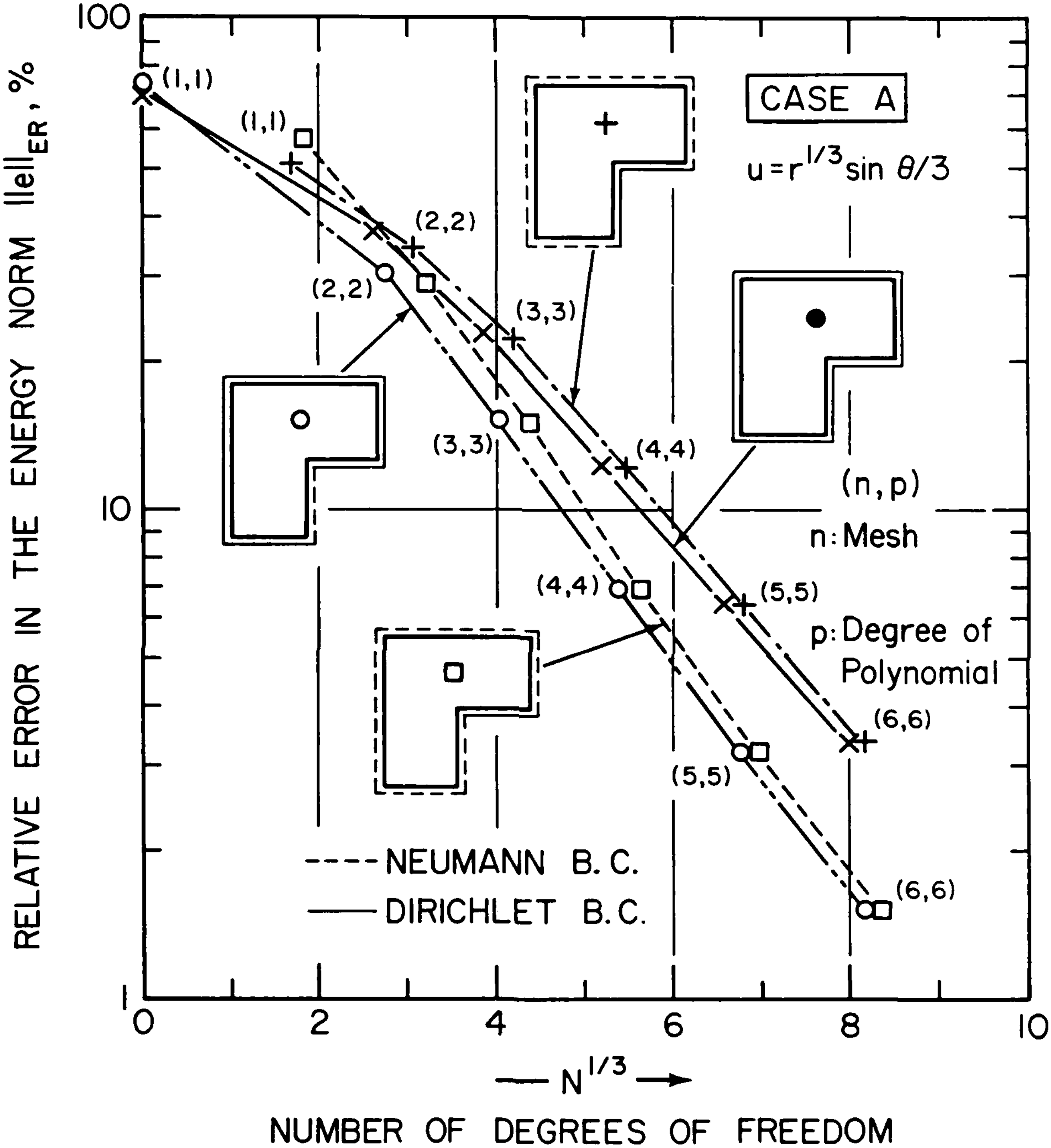

Figure 7.6a. The relative error of the h-p version for various boundary conditions, Case A.

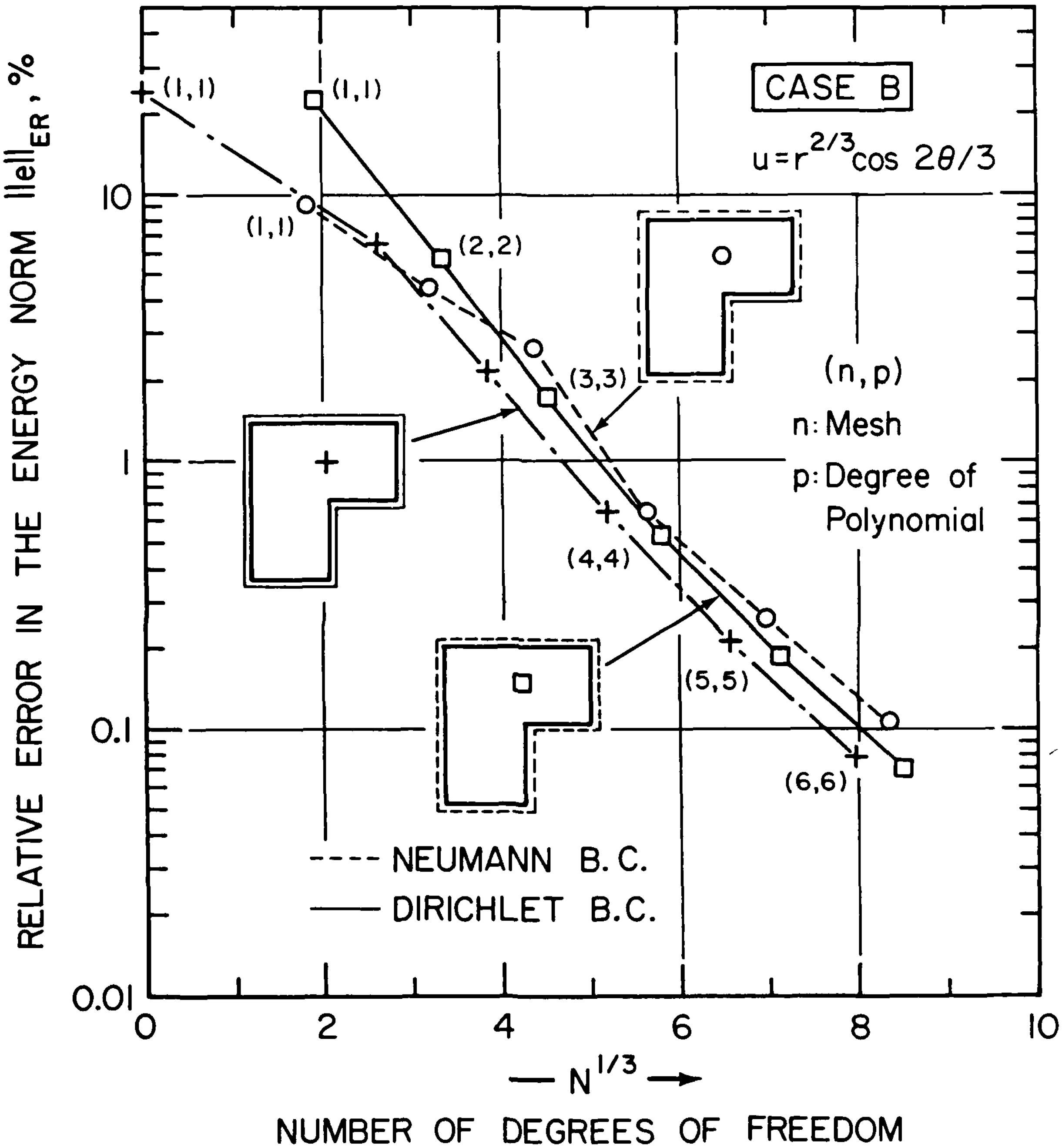

Figure 7.6b. The relative error of the h-p version for various boundary conditions, Case B.

In Figures 7.6a,b we show the performance of the h-p version ($p = n$) for various combinations of the Dirichlet and Neumann conditions (using H^1-projection technique for Dirichlet conditions). We see that there is no significant difference between the performance of the h-p version for various combinations of the boundary conditions. (We mention that in the case A the Dirichlet condition on Γ_1 is homogeneous and so it does not contribute to the error.) In contrast the p-version with the $H^1(\Gamma)$ projection performs independently of the boundary conditions only if $g_i \in H^1(\Gamma_i)$ while for the Dirichlet condition with $g_i \in H^{\sigma}(\Gamma_i)$, $\sigma < 1$ the performance deteriorates. This can be seen by comparison of Figures 7.3a,b resp. 7.7a,b where the error is given for the Dirichlet resp. Neumann boundary condition. We see that in the case B the performance of the p-version (with H^1 projection) for Dirichlet boundary conditions is the same as for the Neumann condition while in the case A we see significant differences.

If the Neumann condition or Neumann condition and homogeneous Dirichlet conditions is prescribed, then the strain energy of the finite element solution is increasing with p, i.e., $E_{FE}(n,p_1) \le E_{FE}(n,p_2)$ for $p_2 \ge p_1$. Because increasing n, the shape of elements is changed, we do not have necessarily $E_{FE}(n_1,p) \le E_{FE}(n_2,p)$ for $n_2 > n_1$ although practically this usually occurs. If the continuous Dirichlet condition is prescribed on the entire boundary and g_i are polynomials of degree $p \ge p_0$ then $E_{FE}(n,p_2) \le E(n,p_1)$ for $p_2 \ge p_1 \ge p_0$, i.e., the strain energy is decreasing with p. If the Dirichlet condition is not a polynomial or on a part of $\partial\Omega$ the nonhomogeneous Neumann condition is prescribed while the Dirichlet condition is given on the other part, the energy is not monotonic.

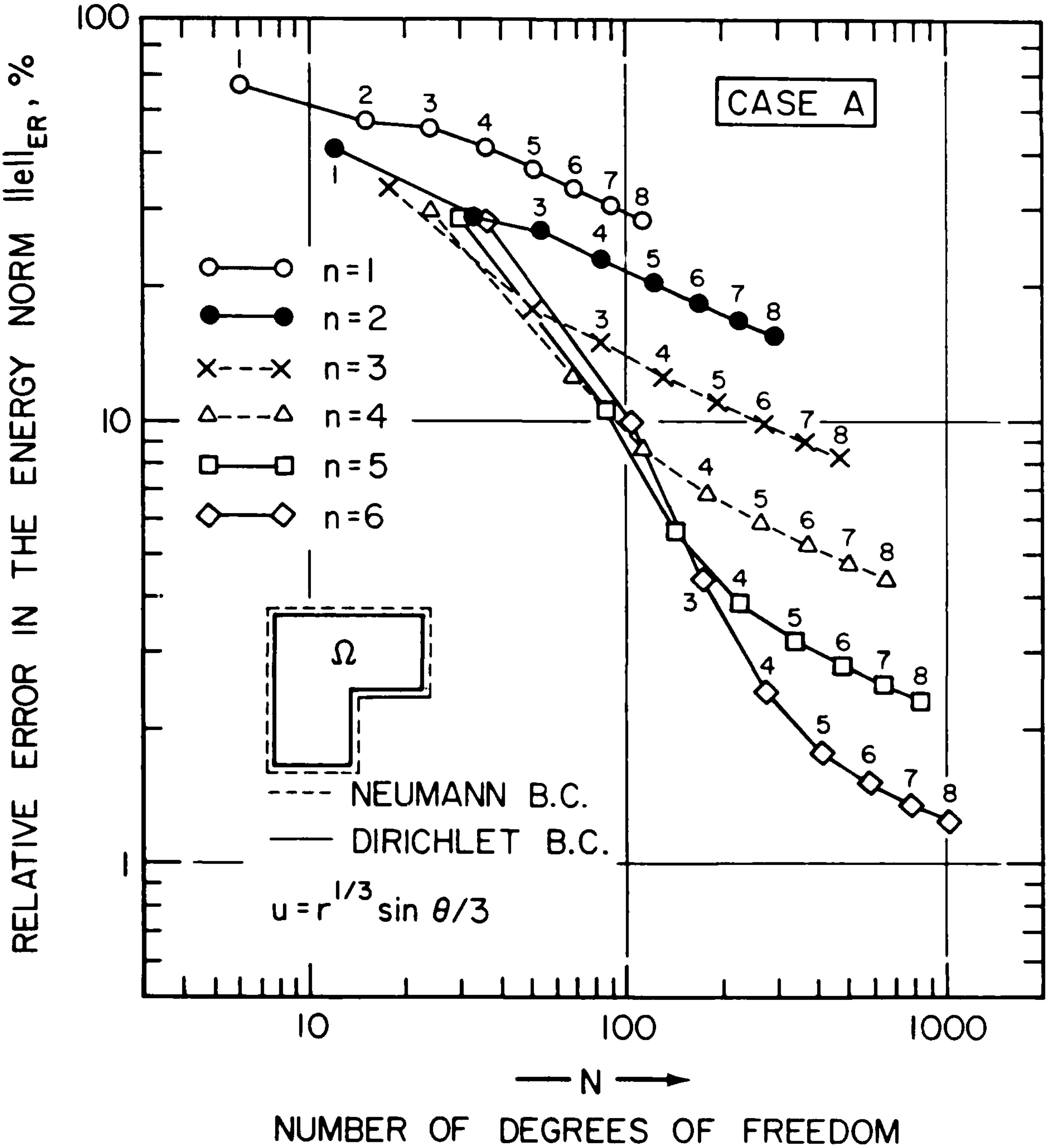

Figure 7.7a. The relative error of the p-version for the Neumann boundary conditions in $\ell g\ \|e\|_{ER} - \ell g\ N$, Case A.

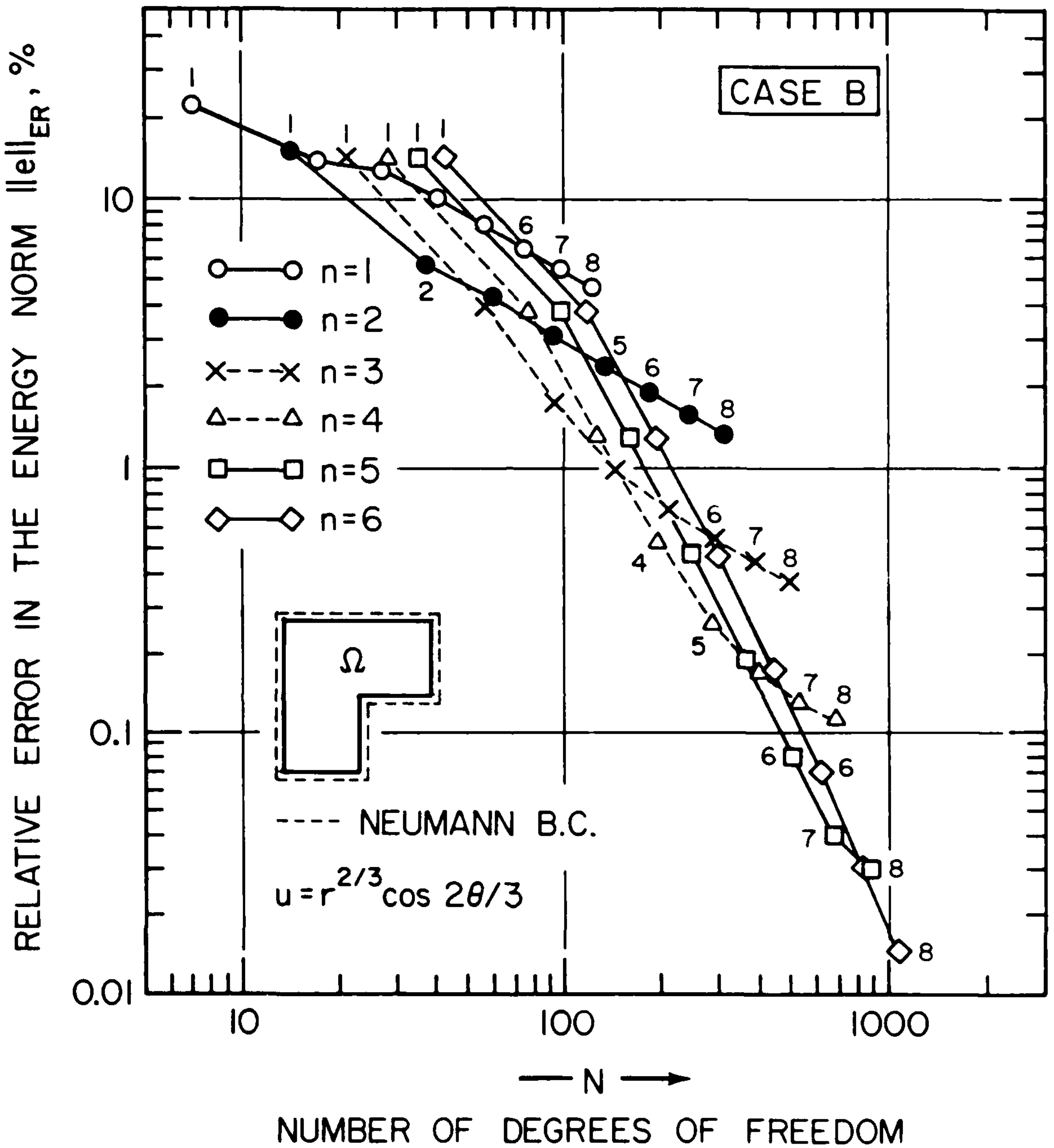

Figure 7.7b. The relative error of the p-version for the Neumann boundary conditions in ℓg $\|e\|_{ER}$ · ℓg N, Case B.

Figure 7.8 shows the behavior of E_{FE} for the case A, where $n = 1,2$ and $p = 1,\ldots,8$ when the monotonicity occurs only in the case d as expected.

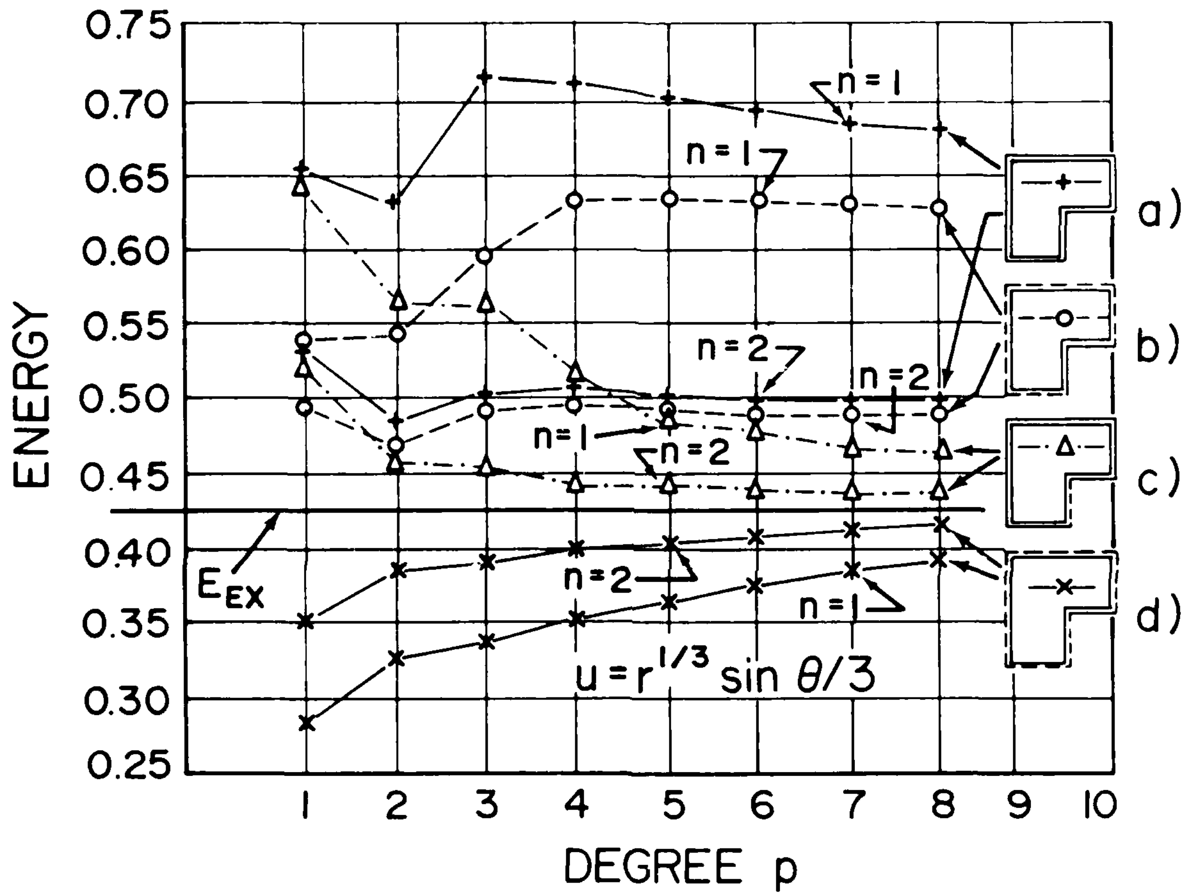

Figure 7.8. The monotonicity of the energy of the finite element solutions.

In the case when the approximation of the nonhomogeneous Dirichlet boundary conditions do not contribute to the error we have

$$\|e\|_E^2 = |E(u_S) - E(u)|.$$

In the case when the Dirichlet conditions are prescribed on the entire $\partial\Omega$ we have

$$\|e\|_E^2 = |(E(u_S) - E(u)) + \mathcal{R}|$$

where the correction term $\mathcal{R}$ is due to the fact that the finite element solution does not satisfy exactly the boundary condition (i.e., $g_S \neq g$). Nevertheless $\mathcal{R}$ is usually negligible for $p \geq 2$ and $|E(u_S) - E(u)|^{1/2} = \|e\|_{\tilde{E}}$ is very close to $\|e\|_E$. The term $\mathcal{R}$ can be easily computed if the small solution is known. In fact

$$\begin{aligned}
\|e\|_E^2 = \|u-u_S\|^2 &= \tfrac{1}{2}B(u_S-u, u_S-u) \\
&= \tfrac{1}{2}\left[B(u_S,u_S) + B(u,u) - 2B(u,u_S)\right] \\
&= \tfrac{1}{2}\left[B(u_S,u_S) - B(u,u)\right] + B(u,u-u_S) \\
&= E(u_S) - E(u) + \mathcal{R}
\end{aligned}$$

where

$$\mathcal{R} = B(u,u-u_S).$$

Because $\Delta u = 0$ we have by integrating by parts

$$\mathcal{R} = \int_{\partial\Omega} \frac{\partial u}{\partial n}\,(u-u_S)\,ds$$

and $u-u_S$ is known on $\partial\Omega$ as well as $\frac{\partial u}{\partial n}$. Table 7.1a,b shows the correction term $\mathcal{R}$ for the mesh Ω_6^n, $n = 6$ and the relative value of $\mathcal{R}/\|e\|_{\tilde{E}}$ depending on p for the case A and B.

Table 7.1a. The correction term $\mathcal{R}$

CASE A

p	$\mathcal{R}$	$\|\underset{\sim}{e}\|_E^2$	$\mathcal{R}/\|\underset{\sim}{e}\|_E^2$ %
1	-1.237(-2)	5.626(-2)	21.99(0)
2	-1.755(-4)	4.945(-3)	3.55 (0)
3	-4.229(-6)	3.419(-3)	1.21(-1)
4	-1.203(-7)	6.091(-4)	1.97(-2)
5	1.053(-11)	5.027(-4)	2.09(-6)
6	3.927(-9)	4.701(-4)	8.35(-4)
7	4.064(-9)	4.535(-4)	8.96(-4)
8	4.069(-9)	4.433(-4)	9.18(-4)

Table 7.3b. The correction term $\mathcal{R}$

CASE B

p	$\mathcal{R}$	$\|\underset{\sim}{e}\|_E^2$	$\mathcal{R}/\|\underset{\sim}{e}\|_E^2$ %
1	-2.698(-2)	5.082(-2)	19.67(0)
2	-1.542(-4)	1.695(-3)	9.09 (0)
3	-2.614(-6)	1.801(-4)	1.44 (0)
4	-6.126(-8)	2.296(-5)	2.67(-1)
5	-1.629(-9)	3.336(-6)	5.04(-2)
6	-4.233(-11)	5.512(-7)	7.68(-3)
7	7.636(-12)	1.202(-7)	6.35(-3)
8	9.268(-12)	4.619(-8)	2.01(-3)

Let us mention that for higher p the correction is influenced by round off errors.

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- To provide a limited consulting service in all areas of numerical mathematics to the University as a whole, and also to government agencies and industries in the State of Maryland and the Washington Metropolitan area.

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- To be an international center of study and research for foreign students in numerical mathematics who are supported by foreign governments or exchange agencies (Fulbright, etc.)

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